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ON A TWO-PARAMETER FAMILY OF NONHOMOGENEOUS MEAN VALUES

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Abstract. In the article, a two-parameter family of nonhomogeneous means is considered, and its basic properties and monotonicity are investigated. This paper is dedicated to my advisor, Prof. Yi-Pei Chen, at Ximaen University.

1. Introduction

The history of mean values is long. One finds the history in [2, 7]. A survey of some recent developments can be found in [8, 12, 13]. The mean values are related to the Mean Value Theorem for derivative and for integral, which is the bridge between the local and global properties of functions. Inequalities of mean values are the main part of theory of analytic inequalities, they have explicit geometric meanings [16].

The simplest and classical means are the arithmetic mean or average, $A(x,y) = (x+y)/2$, the geometric mean or mean proportional, $G(x,y) = \sqrt{xy}$, and the harmonic mean, $H = G^2/A$. They have been generalised, extended, variegated, and refined to a lot of forms. The root-mean-square is defined as $N = (G+A)/2$ and the power means or Hölder means as $M_r(x,y) = ((x^r + y^r)/2)^{1/r}$, $r \neq 0$, $M_0(x,y) = G(x,y)$. In this paper, the variables x and y are positive.

Further evolution led to multivariable means with $(x_1, x_2, \dots, x_n)$ replacing (x,y), to abstracted means $M_\varphi = \varphi^{-1}((\varphi(x) + \varphi(y))/2)$ which reduce to M_r when $\varphi(x) = x^r$, to weighted means which are given by $(1-\alpha)x + \alpha y$ and $x^{1-\alpha}y^\alpha$, $0 \leq \alpha \leq 1$, to Lehmer means $L_p(x,y) = (x^p + y^p)/(x^{p-1} + y^{p-1})$, $p > 0$, which reduce to anti-harmonic mean $L_2(x,y) = (x^2+y^2)/(x+y)$.

Along with means M_r there are more extended means of particular interest. Pólya and Szegö in [15] defined the logarithmic mean L by

$$L = L(x,y) = (x-y)/(\ln x - \ln y) \tag{1}$$

for $x > 0$, $y > 0$ and $x \neq y$, and $L(x,x) = x$.

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Galvani in [6] considered the extended logarithmic means or Stolarsky's means:

$$S_p(x,y)=\left(\frac{y^p-x^p}{p(y-x)}\right)^{1/(p-1)}, \quad x\neq y,\ p\neq 0,1; \tag{2}$$

and $S_p(x,x)=x$; which is reduced to $S_0(x,y)=L(x,y)$, and to the identric mean of the exponential mean $I(x,y)$:

$$S_1(x,y)=I(x,y)=e^{-1}(x^x/y^y)^{1/(x-y)}, \quad x\neq y; \tag{3}$$

and $S_1(x,x)=I(x,x)=x$.

They symmetric means $Q_p(x,y)$ is also defined by

$$Q_p(x,y)=(x^ry^s+x^sy^r)/2 \tag{4}$$

where $r=(1+\sqrt{p})/2$, $s=(1-\sqrt{p})/2$, $p\geq 0$.

Ren-er Yang and Dong-ji Cao in [24] and Horst Alzer in [1] generalized $L(x,y)$ to the one-parameter means:

$$\begin{aligned}
J_p(x,y)&=\frac{p(y^{p+1}-x^{p+1})}{(p+1)(y^p-x^p)}, \quad x\neq y, p\neq 0,-1; \\
J_0(x,y)&=L(x,y), \quad J_{-1}(x,y)=G^2/L; \\
J_p(x,x)&=x.
\end{aligned} \tag{5}$$

Here, $J_{1/2}(x,y)=h(x,y)$ is called Heron mean and $J_2(x,y)$ the centroidal mean.

Ji Chen and Hai-bing Shu [5] introduced the extended Heron means $h_n(x,y)$ by

$$h_n(x,y)=\frac{1}{n+1}\sum_{k=0}^{n}x^{1-k/n}y^{k/n}, \tag{6}$$

and they verified that h_n is a decreasing sequence.

Stolarsky in [20] defined a two-parameter family of extended means $E(r,s;x,y)$:

$$\begin{aligned}
E(r,s;x,y)&=\left(\frac{r}{s}\cdot\frac{y^s-x^s}{y^r-x^r}\right)^{1/(s-r)}, \quad rs(r-s)(x-y)\neq 0; \\
E(r,0;x,y)&=E(0,r;x,y)=L_r(x,y)=(L(x^r,y^r))^{1/r}, \quad r(x-y)\neq 0; \\
E(r,r;x,y)&=I_r(x,y)=(I(x^r,y^r))^{1/r}, \quad x-y\neq 0; \\
E(0,0;x,y)&=G(x,y), \quad x\neq y; \\
E(r,s;x,x)&=x, \quad x=y.
\end{aligned} \tag{7}$$

He showed that E can be extended to be continuous on the domain

$$\{(r,s;x,y): r,s\in R, x,y>0\}.$$

A function similar to E that involves a transformation of values of (r,s) was given by Cisbani [4] and by Tobey [23].

Toader in [21, 22] considered general means

$$M_{r,s}(x,y) = \Big(C_{rs} \cdot \frac{f_r(x,y)}{g_s(x,y)}\Big)^{1/(r-s)}, \tag{8}$$

where f_r and g_s are homogeneous functions of degree r and s, respectively, and $C_{rs} = \lim_{t\to 1}(g_s(1,t)/f_r(1,t))$.

The author also researched the mean values in [16, 18] and the extended means E in [17, 19] by a simpler method.

It is easy to see that these particular means above are special cases of means introduced by Tobey [23]. The study of these means has a rich literature, e.g., for details see [3, 8, 9, 10, 11, 12, 13, 14, 16, 17].

We introduce below a more complicated two-parameter family of nonhomogeneous means $E_{2n}(r,s;x,y)$, and give their basic properties and basic results concerning monotonicity, comparability, and the like.

Study of E_{2n} is interesting, both because most of the two-variable means stated above are special cases of E_{2n}, and because it is challenging to study a function whose formulation is so indeterminate.

2. Definitions and Basic Properties

Let $U_n(t;u)$ be a sequence satisfying

$$t\partial U_n(t;u)/\partial t - (n+1)U_n(t;u) = U_{n+1}(t;u), \tag{9}$$

and $U_0(t;u) = u^t$ for $u > 0$, $n \in N$.

Definition 1. The extended logarithmic means of $2n$-th order $L_{2n}(r;x,y)$ are a one-parameter family of nonhomogeneous means defined, for $n \in N$, as

$$L_{2n}(r;x,y) = \Big[\frac{2n+1}{r^{2n+1}} \cdot \frac{U_{2n}(r;y) - U_{2n}(r;x)}{(\ln y)^{2n+1} - (\ln x)^{2n+1}}\Big]^{1/r}, \quad r(x-y) \neq 0; \tag{10}$$

$$L_{2n}(0;x,y) = \exp\Big[\frac{2n+1}{2n+2} \cdot \frac{(\ln y)^{2n+2} - (\ln x)^{2n+2}}{(\ln y)^{2n+1} - (\ln x)^{2n+1}}\Big], \quad x \neq y; \tag{11}$$

$$L_{2n}(r;x,x) = x, \quad r \in R.$$

Definition 2. The extended means of $2n$-th order $E_{2n}(r,s;x,y)$ are a two-parameter family of nonhomogeneous means defined, for $n \in N$, by

$$E_{2n}(r,s;x,y) = \Big[\frac{r^{2n+1}}{s^{2n+1}} \cdot \frac{U_{2n}(s;y) - U_{2n}(s;x)}{U_{2n}(r;y) - U_{2n}(r;x)}\Big]^{1/(s-r)}, \quad rs(s-r)(x-y) \neq 0; \tag{12}$$

$$E_{2n}(r,r;x,y) = \exp\Big[\frac{1}{r} \cdot \frac{U_{2n+1}(r;y) - U_{2n+1}(r;x)}{U_{2n}(r;y) - U_{2n}(r;x)}\Big], \quad r(x-y) \neq 0; \tag{13}$$

$$E_{2n}(r,0;x,y) = L_{2n}(r;x,y), \quad x \neq y, r \in R;$$

$$E_{2n}(r,s;x,x) = x, \quad r,s \in R.$$

For the sake of convenience, we write $E_{2n}(r,s;x,y)=E_{2n}(r,s)=E_{2n}(x,y)=E_{2n}$, shifting notation to suit the context.

Theorem 1. *$L_{2n}(r;x,y)$ and $E_{2n}(r,s;x,y)$ can be expressed in integral forms:*

$$L_{2n}(r;x,y)=E_{2n}(r,0;x,y)=\left[\frac{\int_x^y(\ln u)^{2n}u^{r-1}du}{\int_x^y(\ln u)^{2n}u^{-1}du}\right]^{1/r},\quad r(x-y)\neq 0; \tag{14}$$

$$L_{2n}(0;x,y)=\exp\left(\frac{\int_x^y(\ln u)^{2n+1}u^{-1}du}{\int_x^y(\ln u)^{2n}u^{-1}du}\right),\quad x\neq y; \tag{15}$$

$$E_{2n}(r,s;x,y)=\left(\frac{\int_x^y(\ln u)^{2n}u^{s-1}du}{\int_x^y(\ln u)^{2n}u^{r-1}du}\right)^{1/(s-r)},\quad (r-s)(x-y)\neq 0; \tag{16}$$

$$E_{2n}(r,r;x,y)=\exp\left(\frac{\int_x^y(\ln u)^{2n+1}u^{r-1}du}{\int_x^y(\ln u)^{2n}u^{r-1}du}\right),\quad x\neq y. \tag{17}$$

Proof. Let $g(t;x,y)=(y^t-x^t)/t$ for $t\neq 0$ and $g(0;x,y)=\ln y-\ln x$. By direct computation and induction on n, it follows that

$$g_t^{(n)}(t;x,y)=(U_n(t;y)-U_n(t;x))/t^{n+1}, \tag{18}$$

$$\partial U_n(t;u)/\partial u=(\ln u)^n u^{t-1}t^{n+1}, \tag{19}$$

$$g_t^{(n)}(t;x,y)=\int_x^y(\ln u)^n u^{t-1}du. \tag{20}$$

This implies Theorem 1.

Corollary 1. *$L_{2n}(r;x,y)$ is continuous on the domain*

$$\{(r;x,y)|x>0,y>0,r\in R\}$$

and $E_{2n}(r,s;x,y)$ is continuous on the domain

$$\{(r,s;x,y)|x>0,y>0,r,s\in R\}.$$

Theorem 2. *$L_{2n}(r;x,y)$ and $E_{2n}(r,s;x,y)$ have the following properties:*

$$\begin{aligned}
&\min\{x,y\}\le L_{2n}(r;x,y)\le\max\{x,y\},\\
&L_{2n}(r;x,y)=L_{2n}(r;y,x),\\
&L_0(r;x,y)=L_r(x,y),\\
&L_{2n}(r;x^\alpha,y^\alpha)=(L_{2n}(\alpha r;x,y))^\alpha,\quad \alpha\neq 0;\\
&\min\{x,y\}\le E_{2n}(r,s;x,y)\le\max\{x,y\},\\
&E_{2n}(r,s;x,y)=E_{2n}(r,s;y,x)=E_{2n}(s,r;x,y),\\
&E_0(r,s;x,y)=E(r,s;x,y),\\
&E_0(r,r;x,y)=I_r(x,y),\\
&[E_{2n}(r,t)]^{r-t}=[E_{2n}(r,s)]^{r-s}[E_{2n}(s,t)]^{s-t},\\
&[E_{2n}(\alpha r,\alpha s;x,y)]^\alpha=E_{2n}(r,s;x^\alpha,y^\alpha),\quad \alpha\neq 0.
\end{aligned}$$

Proof. They follow from the mean value theorem for integral, Theorem 1 and standard arguments.

Lemma 1. *Let $f, h : [a,b] \to R$ be integrable functions, both increasing or both decreasing. Furthermore, let $p : [a,b] \to R^+$ be an integrable function. Then*

$$\int_a^b p(u)f(u)du \int_a^b p(u)h(u)du \le \int_a^b p(u)du \int_a^b p(u)f(u)h(u)du. \tag{21}$$

If one of the functions of f or h is nonincreasing and the other nondecreasing, then the inequality (21) *reverses.*

Inequality (21) is called the Tchebycheff integral inequality. For proof of it, see [2, 7, 8, 12, 13, 18].

Proposition 1. *Let $g = g(t;x,y) = \int_x^y u^{t-1}du$, $x \ne y$. Then, for k, $j \in N$, we have*

$$g_t^{(2k+1)} g_t^{(2(j+k)+1)} \le g_t^{(2k)} g_t^{(2(j+k+1))}. \tag{22}$$

The ratio $g_t^{(2(j+k)+1)}(t;x,y)/g_t^{(2k)}(t;x,y)$ is increasing in t.

Proof. Inequality (22) is a special case of the Tchebycheff (or Čebyšev) integral inequality applied to the functions $p(u) = (\ln u)^{2k}u^{t-1}$, $f(u) = \ln u$ and $h(u) = (\ln u)^{2j+1}$ for j, $k \in N$, $t \in R$ and $u \in [x,y]$.

Inequality (22) and direct calculation produce

$$\left(\frac{g_t^{(2(j+k)+1)}}{g_t^{(2k)}}\right)_t = \frac{g_t^{(2(j+k+1))} g_t^{(2k)} - g_t^{(2(j+k)+1)} g_t^{(2k+1)}}{(g_t^{(2k)})^2} \ge 0.$$

Therefore, the desired result follows.

Theorem 3. *$L_{2n}(r;x,y)$ increases with respect to r.*

Proof. From Theorem 1 we have

$$\begin{aligned}
[\ln L_{2n}(r;x,y)]_t &= \frac{1}{r^2}\left(\frac{\int_x^y (\ln u)^{2n+1}u^{r-1}du}{\int_x^y (\ln u)^{2n}u^{r-1}du} \cdot r - \ln \frac{\int_x^y (\ln u)^{2n}u^{r-1}du}{\int_x^y (\ln u)^{2n}u^{-1}du}\right) \\
&= \frac{1}{r}\left(\frac{\int_x^y (\ln u)^{2n+1}u^{r-1}du}{\int_x^y (\ln u)^{2n}u^{r-1}du} - \frac{\int_x^y (\ln u)^{2n+1}u^{\theta-1}du}{\int_x^y (\ln u)^{2n}u^{\theta-1}du}\right),
\end{aligned}$$

where θ is between 0 and r, by the mean value theorem for derivative. From Proposition 1, $[\ln L_{2n}(r;x,y)]_t > 0$, therefore $L_{2n}(r;x,y)$ is increasing with respect to r for $n \in N$.

Theorem 4. *If $x, y > 1$, then*

$$L_{2n}(r;x,y) = E_{2n}(r,0;x,y) \le E_{2n+2}(r,0;x,y) = L_{2n+2}(r;x,y). \tag{23}$$

If $0 < x < y < 1$, inequality (23) *is reversed.*

Proof. Using formulae (14) and (15), the Tchebycheff integral inequality applied to $p(u) = u^{-1}(\ln u)^{2n}$, $f(u) = u^r$ and $h(u) = (\ln u)^2$, and the standard arguments result in Theorem 4.

Theorem 5. *$L_{2n}(r;x,y)$ increases in both x and y.*

Proof. From (15) and (18), direct computation produces

$$\begin{aligned}\ln L_{2n}(0;x,y) &= g_r^{(2n+1)}(0;x,y)/g_r^{(2n)}(0;x,y),\\ \frac{\partial[\ln L_{2n}(0;x,y)]}{\partial x} &= \Big[\int_x^y (\ln u)^{2n+1}u^{-1}du\\ &\quad -(\ln x)\int_x^y (\ln u)^{2n}u^{-1}du\Big]\frac{(\ln x)^{2n}x^{-1}}{[g_r^{(2n)}(0;x,y)]^2} \geq 0.\end{aligned}$$

Hence, $L_{2n}(0;x,y)$ increases with respect to both x and y.

Since $L_{2n}^r(r;x,y) = g_r^{(2n)}(r;x,y)/g_r^{(2n)}(0;x,y)$, easy calculation results in

$$\begin{aligned}\frac{\partial L_{2n}^r(r;x,y)}{\partial x} &= [g_r^{(2n)}(r;x,y) - x^r g_r^{(2n)}(0;x,y)]x^{-1}(\ln x)^{2n}/[g_r^{(2n)}(0;x,y)]^2\\ &= \Big[\int_x^y (\ln u)^{2n}u^{r-1}du - x^r\int_x^y (\ln u)^{2n}u^{-1}du\Big]\frac{(\ln x)^{2n}x^{-1}}{[g_r^{(2n)}(0;x,y)]^2},\end{aligned}$$

$$\operatorname{sgn}(\partial L_{2n}^r(r;x,y)/\partial x) = \operatorname{sgn} r.$$

Therefore, $L_{2n}(r;x,y)$ increases with both x and y.

4. Monotonicity of $E_{2n}(r,s;x,y)$

Theorem 6. *$E_{2n}(r,s;x,y)$ increases with respect to both r and s.*

Proof. From (17) and Proposition 1, it follows that $E_{2n}(r,r;x,y)$ increases with both r and s.

By easy computation, we have

$$[\ln E_{2n}(r,s;x,y)]_s = \frac{1}{(s-r)^2}\left[\frac{g_s^{2n+1}(s;x,y)}{g_s^{(2n)}(s;x,y)}(s-r) - \ln\frac{g_s^{(2n)}(s;x,y)}{g_r^{(2n)}(r;x,y)}\right].$$

By the mean value theorem, we obtain

$$\ln\frac{g_s^{(2n)}(s;x,y)}{g_r^{(2n)}(r;x,y)} = \frac{g_\gamma^{(2n+1)}(\gamma;x,y)}{g_\gamma^{(2n)}(\gamma;x,y)}(s-r) < \frac{g_s^{(2n+1)}(s;x,y)}{g_s^{(2n)}(s;x,y)}(s-r),$$

where γ is between r and s. Therefore, $[\ln E_{2n}(r,s;x,y)]_s > 0$, $E_{2n}(r,s;x,y)$ is increasing in s. Since $E_{2n}(r,s;x,y) = E_{2n}(s,r;x,y)$, it is deduced that $E_{2n}(r,s;x,y)$ increases in both r and s.

Theorem 7. *For $x, y > 1$,*

$$E_{2n}(r,s;x,y) \le E_{2n+2}(r,s;x,y). \tag{24}$$

If $0 < x < y < 1$, inequality (24) reverses.

Proof. For $s > r$, inequality (24) is equivalent to

$$\frac{g_s^{(2n+2)}(s;x,y)}{g_r^{(2n+2)}(r;x,y)} \ge \frac{g_s^{(2n)}(s;x,y)}{g_r^{(2n)}(r;x,y)},$$

that is

$$\frac{\int_x^y (\ln u)^{2n+2} u^{s-1} du}{\int_x^y (\ln u)^{2n+2} u^{r-1} du} \ge \frac{\int_x^y (\ln u)^{2n} u^{s-1} du}{\int_x^y (\ln u)^{2n} u^{r-1} du}.$$

From (21), applied to $p(u) = (\ln u)^{2n} u^{r-1}$, $f(u) = (\ln u)^2$, $g(u) = u^{s-r}$, the inequality (24) follows.

Theorem 8. *$E_{2n}(r,s;x,y)$ increases with both x and y.*

Proof. Since $\ln E_{2n}(r,r;x,y) = g_r^{(2n+1)}(r;x,y)/g_r^{(2n)}(r;x,y)$, calculating straightforwardly yields

$$[\ln E_{2n}(r,r;x,y)]_x = \left[g_r^{(2n+1)}(r;x,y) - (\ln x) g_r^{(2n)}(r;x,y)\right] \frac{x^{r-1}(\ln x)^{2n}}{[g_r^{(2n)}(r;x,y)]^2}$$
$$= \left[\int_x^y (\ln u)^{2n+1} u^{r-1} du - (\ln x)\int_x^y (\ln u)^{2n} u^{r-1} du\right] \frac{x^{r-1}(\ln x)^{2n}}{[g_r^{(2n)}(r;x,y)]^2} \ge 0.$$

Thus, $E_{2n}(r,r;x,y)$ is increasing in both x and y.

Let

$$P_{2n}(r,s;x,y) = [E_{2n}(r,s;x,y)]^{s-r} \cdot \frac{s^{2n+1}}{r^{2n+1}} = \frac{U_{2n}(s;y) - U_{2n}(s;x)}{U_{2n}(r;y) - U_{2n}(r;x)}.$$

Computating directly arrives at

$$\frac{\partial P_{2n}}{\partial x} = \frac{(\ln x)^{2n}(rs)^{2n+1} x^{r+s-1}}{[U_{2n}(r;y) - U_{2n}(r;x)]^2}\left[\frac{g_r^{(2n)}(r;y,x)}{x^r} - \frac{g_s^{(2n)}(s;y,x)}{x^s}\right],$$

$$\left(\frac{g_t^{(2n)}(t;y,x)}{x^t}\right)_t = \frac{g_t^{(2n+1)}(t;y,x) - (\ln x) g_t^{(2n)}(t;y,x)}{x^t}$$
$$= \frac{\int_y^x (\ln u)^{2n+1} u^{t-1} du - (\ln x)\int_y^x (\ln u)^{2n} u^{t-1} du}{x^t}.$$

From this we conclude that

$$\operatorname{sgn}\left(\frac{\partial E_{2n}^{s-r}}{\partial x}\right) = \operatorname{sgn}\left(\frac{r^{2n+1}}{s^{2n+1}} \cdot \frac{\partial P_{2n}}{\partial x}\right)$$
$$= \operatorname{sgn}\left(\frac{g_r^{(2n)}(r;y,x)}{x^r} - \frac{g_s^{(2n)}(s;y,x)}{x^s}\right) = \operatorname{sgn}(s-r).$$

Therefore, $\partial E_{2n}(r,s;x,y)/\partial x > 0$, $E_{2n}(r,s;x,y)$ is increasing with both x and y.

Remark. Some properties of the function $g(t;x,y)$ had been given in [16, 17, 18, 19].

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