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ON BOREL DIRECTION CONCERNING SMALL FUNCTIONS

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Abstract. In this paper, we shall prove

Theorem 1. *Let f be nonconstant meromorphic in $\mathbb{C}$ with finite positive order λ, $\lambda(r)$ be a proximate order of f and $U(r,f)=r^{\lambda(r)}$, then for each number $\alpha, 0<\alpha<\pi/2$, there exists a number φ_0 with $0\le\varphi_0<2\pi$ such that the inequality*

$$\limsup_{r\to+\infty}\sum_{i=1}^{3} n(r,\varphi_0,\alpha,f=a_i(z))/U(r,f)>0,$$

holds for any three distinct meromorphic functions $a_i(z)(i=1,2,3)$ with $T(r,a_i)=o(U(r,f))$, as $r\to+\infty$.

1. Introduction and Main Results

Let f be a function meromorphic in the finite complex plane $\mathbb{C}$. We donote by $T(r,f)(T_0(r,f))$ the Nevanlinna(Ahlfors-Shmizu) characteristic function of f. A meromorphic function $a(z)$ (including the case $f(z)\equiv c$ where c in $\mathbb{C}\cup\{\infty\}$) is called *small* with respect to f if $T(r,a(z))=o(T(r,f))$ as $r\to+\infty$. We let $n(r,\varphi,\alpha,f=a(z))$ be the number of roots (multiple roots being counted with their multiplicities) of the equation $f(z)=a(z)$ for z in the angular domain $\Omega(r,\varphi,\alpha)=\{z:|argz-\varphi|<\alpha,|z|<r\}$ where $0\le\varphi<2\pi,\alpha>0$.

This paper deals with the existence of the Borel directions concerning small functions for mermorphic functions of finite positive order. Using Tsuji's method, we shall mainly prove Theorem 1 stated in the abstract. Theorem 1 extends a result of Chuang [2, p.127, Corollary 5.3], there $a(z)$ are restricted over all extended complex numbers. Chuang's method is different from ours and is based on the existence of a sequence of filling disk with their roots in the works of Milloux [3] and Valiron [7].

Theorem 2. (The Existence Theorem on Borel Direction concerning small functions) *If f is meromorphic in $\mathbb{C}$ with order λ, $0<\lambda<+\infty$, then there exists a number φ_0 with*

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$0 \leq \varphi_0 < 2\pi$, *such that for each* $\alpha > 0$, *the equation*

$$\limsup_{r \to +\infty} \log\{\sum_{i=1}^{3} n(r, \varphi_0, \alpha, f = a_i(z))\} / \log r = \lambda,$$

holds for any three distinct meromorphic functions $a_i(z)$ $(i = 1, 2, 3)$ *with* $T(r, a_i) = o(T(r, f))$, *as* $r \to +\infty$.

Theorem 2 extends a result of Biernacki [1, Theorem 6.5].

Applying Theorem 1 and adapting a line of reasoning used by C. T. Chuang [2, p.128, Corollary 5.4], it is easy to obtain the following

Theorem 3. [4] *If* f *is meromorphic in* $\mathbb{C}$ *with finite positive order,* $U(r, f)$ *is given as in Theorem* 1, *then there is a number* φ_0 *with* $0 \leq \varphi_0 < 2\pi$ *such that for each positive number* α, *the inequality*

$$\limsup_{r \to +\infty} \sum_{i=1}^{3} n(r, \varphi_0, \alpha, f = a_i(z))/U(r, f) > 0,$$

holds for any three distinct meromorphic functions $a_i(z)(i = 1, 2, 3)$ *with* $T(r, a_i) = o(U(r, f))$, *as* $r \to +\infty$.

Theorem 3 extends a result of Valiron [8, p.34, Theorem 29.]. Since $\limsup_{r \to +\infty} \frac{T(r,f)}{U(r,f)} = 1$, Theorem 2 follows from Theorein 3.

2. The Proof of Theorem 1

To prove Theorem 1, we need some terminologies. Let $S(r, \varphi, \alpha, f)$ be the spherical area of the image under f of $\Omega(r, \varphi, \alpha)$ where $0 \leq \varphi \leq 2\pi$. $T_0(r, \varphi, \alpha, f)$ be the Ahlfors-Shimizu characteristic of f associated with $S(r, \varphi, \alpha, f)$ and $N(r, \varphi, \alpha, f = a(z))$ be the integral counting function of f associated with $n(r, \varphi, \alpha, f = a(z))$. Suppose that the conclusion of Theorem 1 is incorrect, then there exists a positive number $\alpha, 0 < \alpha < \pi/2$; for each $\varphi, o \leq \varphi < 2\pi$, there exist three distinct meoromorphic functions a_{φ_j} $(j = 1, 2, 3)$ with $T(r, a_{\varphi_j}) = o(U(r, f))$ such that the expression

$$\limsup_{r \to +\infty} \sum_{j=1}^{3} n(r, \varphi_0, \alpha, f = a_{\varphi_j}(z))/U(r, f) = 0, \tag{2.1}$$

holds.

Since $\chi = \{(\varphi - \alpha/4, \varphi + \alpha/4) : \varphi \in [0, 2\pi]\}$ is an open covering of the closed interval $[0, 2\pi]$ and $[0, 2\pi]$ is compact; so there exists a finite subcovering $\chi_0 = \{(\varphi_k - \alpha/4, \varphi_k + \alpha/4) | k = 1, \ldots, n\}$ which covers $[0, 2\pi]$.

For each positive integer $k, 1 \leq k \leq n$, we put

$$F_k(z) = (f(z) - a_{\varphi_{k1}}(z))(a_{\varphi_{k3}}(z) - a_{\varphi_{k2}}(z))/(f(z) - a_{\varphi_{k2}}(z))(a_{\varphi_{k3}}(z) - a_{\varphi_{k1}}(z)), \quad (2.2)$$

where $a_{\varphi_{kj}}(z), (j = 1, 2, 3)$ depending on φ_k and α and satisfying the expression (2.1). The function f can be written as

$$f = (g_{\varphi_{k1}} F_k + g_{\varphi_{k2}}/(g_{\varphi_{k3}} F_k + g_{\varphi_{k4}}). \quad (2.3)$$

For above expression (2.3), applying [6, Lemma, p.277], we have

$$S(r, \varphi_k, \alpha/4, f) \leq 27S(64r, \varphi_k, \alpha/2, F_k) + o\Big(\int_1^{128r} (\sum_{j=1}^{4} T(t, g_{\varphi_{kj}})/t)dt\Big). \quad (2.4)$$

Dividing two sides of the above inequality (2.4) by r, and then integrating them to r, and then applying [6, Theorem VII.8, p.272], we have

$$\begin{aligned} T_0(r, \varphi_k, \alpha/4, f) &\leq 27T_0(64r, \varphi_k, \alpha/2, F_k) + o(U(r, f)) \\ &\leq 81 \sum_{j=1}^{3} N(128r, \varphi_k, \alpha, F_k = b_j) + o(U(r, f)) \end{aligned} \quad (2.5)$$

where $b_i = 0, b_2 = 1, b_3 = \infty$.

Since χ_0 covers $[0, 2\pi]$, we have

$$\begin{aligned} T_0(r, f) &\leq \sum_{k=1}^{n} T_0(r, \varphi_k, \alpha/4, f) \\ &\leq 81 \sum_{k=1}^{n} \sum_{j=1}^{3} N(128r, \varphi_k, \alpha, F_k = b_j) + o(U(r, f)) \\ &= 81 \sum_{k=1}^{n} \sum_{j=1}^{3} N(128r, \varphi_k, \alpha, f = a_{\varphi_{kj}}) + o(U(r, f)). \end{aligned} \quad (2.6)$$

Dividing two sides of the above inequality (2.6) by $U(r, f)$, then taking $\limsup_{r \to +}$, and then applying the L' Hopital Rule we have

$$\begin{aligned} \limsup_{r \to +\infty} T_0(r, f) \ / \ U(r, f) &\leq 81 \sum_{k=1}^{n} \limsup_{r \to +\infty} \sum_{j=1}^{3} N(128r, \varphi_k, \alpha, f = a_{\varphi_{kj}})/U(r, f) \\ &= 81 \sum_{k=1}^{n} (128)^{\lambda} \limsup_{r \to +\infty} \sum_{j=1}^{3} N(r, \varphi_k, \alpha, f = a_{\varphi_{kj}})/U(r, f) \\ &\leq 81(128)^{\lambda} \sum_{k=1}^{n} \limsup_{r \to +\infty} \sum_{j=1}^{3} (1/\lambda) n(r, \varphi_k, \alpha, f = a_{\varphi_{kj}})/U(r, f) = 0. \end{aligned} \quad (2.7)$$

Above result contradicts $\limsup_{r\to+\infty} T(r,f)/U(r,f) = 1$, and $T_0(r,f) \sim T(r,f)$. This completes the proof of Theorem 1.

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