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ON SOME SUBCLASSES OF CLOSE-TO-CONVEX FUNCTIONS OF ORDER α TYPE δ

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Abstract. Let $Q^*_\lambda(\alpha,\delta)$ denote the class of analytic functions f in the unit disc E, with $f(0)=0$, $f'(0)=1$ and satisfying the condition

$$Re\left\{(1-\lambda)\frac{zf'(z)}{g(z)}+\lambda\frac{(zf'(z))'}{g'(z)}\right\}>\alpha,$$

for $z\in E$, g starlike function of order $\delta(0\leq\delta\leq 1)$, $0\leq\alpha\leq 1$ and λ complex with $Re\lambda\geq 0$. It is shown that $Q^*_\lambda(\alpha,\delta)$ with $\lambda\geq 0$ are close-to-convex and hence univalent in E. Coefficient results, an integral representation for $Q^*_\lambda(\alpha,\delta)$ and some other properties of $Q^*_\lambda(\alpha,\delta)$ are discussed. The class $Q^*_\lambda(\alpha,1)$ is also investigated in some detail.

1. Introduction

Denote by S the class of functions f which are analytic and univalent in the unit disc $E\{z:|z|<1\}$ and are given by

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n. \tag{1.1}$$

The subclasses S^* and C of starlike and convex functions respectively are well-known and have been extensively studied, see [6],[2] and [5]. A function $f\in S$ is called a convex function of order δ, $0\leq\delta\leq 1$, if, for $z\in E$,

$$\text{Re}\frac{(zf'(z))'}{f'(z)}>\delta.$$

We denote this class by $C(\delta)$. Also $f\in S$ is a starlike function of order δ, $0\leq\delta\leq 1$ if, for $z\in E$, $\text{Re}\frac{zf'(z)}{f(z)}>\delta$, and call this class $S^*(\delta)$. These two classes were introduced by Robertson [17].

In [10], Libera introduced the class $K(\alpha,\delta)$ of close-to-convex functions of order α type δ. A function f, analytic in E and given by (1.1), belongs to the class $K(\alpha,\delta)$,

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$0 \leq \alpha \leq 1$, $0 \leq \delta \leq 1$, if and only if, there exists a function $g \in S^*(\delta)$ such that, for $z \in E$,

$$\mathrm{Re}\frac{zf'(z)}{g(z)} > \alpha.$$

It is clear that $K(0,0) = K$, the class of close-to-convex univalent functions introduced and studied first by Kaplan in [9]. Noor [15] defined a subclass $K^*(\alpha, \delta)$ of univalent functions as follows. A function f, analytic in E and given by (1.1), is in the class $K^*(\alpha, \delta)$, $0 \leq \alpha \leq 1$, $0 \leq \delta \leq 1$, if and only if, there exists a function $g \in S^*(\delta)$ such that

$$\mathrm{Re}\frac{(zf'(z))'}{g'(z)} > \alpha, \quad z \in E$$

For $\alpha = 0$, $\delta = 0$, the class $K^*(0,0)$ reduces to the class K^* studied in [14].

We now define the following.

Definition 1.1. Let f, given by (1.1), be analytic in E and for λ complex with $Re\lambda \geq 0$, let

$$\mathrm{Re}\left\{(1-\lambda)\frac{zf'(z)}{g(z)} + \lambda\frac{(zf'(z))'}{g'(z)}\right\} > \alpha \quad z \in E,$$

for some $\alpha (0 \leq \alpha \leq 1)$ and $g \in S^*(\delta)$. Then f is said to belong to the class $Q^*_\lambda(\alpha, \delta)$ for $z \in E$.

We note that $Q^*_0(\alpha, \delta) = K(\alpha, \delta)$ and $Q^*_1(\alpha, \delta) = K^*(\alpha, \delta)$.

Let f and g be analytic functions in E with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then the convolution (Hadamard product) of f and g is defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n. \tag{1.2}$$

2. Preliminary Results

Lemma 2.1. [13] *Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$ and $\psi(u, v)$ be a complex-valued function satisfying the conditions*

(i) *$\psi(u, v)$ is continuous in a domain $D \subset \mathbb{C}^2$,*

(ii) *$(1, 0) \in D$ and $\psi(1, 0) > 0$*

(iii) *$Re\psi(iu_2, v_1) \leq 0$ whenever $(iu_2, v_1) \in D$ and $v_1 \leq -\frac{1}{2}(1 + u_2^2)$. If $h(z) = 1 + \sum_{m=2}^{\infty} c_m z^m$ is a function, analytic in E, such that $(h(z), zh'(z)) \in D$ and $Re\psi(h(z), zh'(z)) > 0$ for $z \in E$, then $Re\, h(z) > 0$ in E.*

The following is a special case of a result proved in [7].

Lemma 2.2. *Let p be analytic in E, $p(0) = 1$ and $Re\ p(z) > \alpha$ in E. Then, for $z \in E$, $|z| = r$, we have*

$$Re\frac{zp'(z)}{p(z)} \geq \frac{2(1-\alpha)r}{(1-(1-2\alpha)r)(1+r)}, \quad for\ r \leq r_1,$$

where r_1 is the unique root of the equation

$$(2\alpha - 1)r^4 + 2(1-2\alpha)r^3 - 6\alpha r^2 - 2r + 1 = 0 \tag{2.1}$$

in the interval $(0, 1]$.

This result is sharp.

Lemma-2.3.[16] *If $p(z)$ is analytic in E with $p(0) = 1$ and if λ is a complex number satisfying $Re\ \lambda \geq 0$ then $Re\ \{p(z) + \lambda zp'(z)\} > \alpha$, $(0 \leq \alpha < 1)$ implies $Re\ p(z) > \alpha + (1-\alpha)(2\sigma - 1)$ where σ is given by*

$$\sigma = \sigma(Re\lambda) = \int_0^1 (1 + t^{Re\lambda - 1})dt, \tag{2.2}$$

which is an increasing function of $Re\ \lambda$ and $\frac{1}{2} \leq \sigma < 1$. This estimate is sharp in the sense that the bound cannot be improved.

Lemma 2.4.[20] *If $P(z)$ is analytic in $E, P(0) = 0$ and $Re\ P(z) > \frac{1}{2}$, $z \in E$, then, for any function F, analytic in E, the function $P * F$ takes values in the convex hull of the image of E under F.*

3. Main Results

Theorem 3.1. *Let $f \in Q^*_\lambda(\alpha, \delta)$, $\lambda \geq 0$. Then $f \in K(\gamma, \delta)$ where*

$$\gamma = \frac{2\alpha + \lambda d_1}{2 + \lambda\delta_1} \tag{3.1}$$

and

$$\delta_1 = \frac{\mathrm{Re}h(z)}{|h(z)|^2}, \quad \mathrm{Re}h(z) > \delta. \tag{3.2}$$

Proof. Let $g \in S^*(\delta)$ and set

$$\frac{zf'(z)}{g(z)} = [(1-\gamma)p(z) + \gamma].$$

We see that $p(0) = 1$ and p is analytic in E. Simple calculations yield

$$\left\{(1-\lambda)\frac{zf'(z)}{g(z)} + \lambda\frac{(zf'(z))'}{g'(z)}\right\} - \alpha = \left\{(1-\gamma)p(z) + (\gamma - \alpha) + \lambda(1-\lambda)\frac{zp'(z)}{h(z)}\right\}, \tag{3.3}$$

where Re $h(z) = \text{Re}\frac{zg'(z)}{g(z)} > \delta$, $z \in E$.

Since $f \in Q^*_\lambda(\alpha, \delta)$, $\text{Re}\{(1-\gamma)p(z) + (\gamma - \alpha) + \lambda(1-\gamma)\frac{zp'(z)}{h(z)}\} > 0$ in E. We form the functional $\psi(u, v)$ by choosing $u = p(z)$, $v = zp'(z)$.

Thus

$$\psi(u, v) = (1-\gamma)u + (\gamma - \alpha) + \frac{\lambda(1-\gamma)v}{h}.$$

The first two conditions of Lemma 2.1 are clearly satisfied. We verify the condition (iii) as follows.

$$\begin{aligned}\text{Re}\psi(iu_2, v_1) &= (\gamma - \alpha) + \frac{\lambda(1-\gamma)v_1 \text{Re}h}{|h|^2} \\ &= (\gamma - \alpha) + \lambda(1-\gamma)v_1\delta_1, \quad \left(\delta_1 = \frac{\text{Re}h}{|h|^2}\right).\end{aligned}$$

Now, for $v_1 \leq -\frac{1}{2}(1 + u_2^2)$, we have

$$\begin{aligned}\text{Re}\psi(iu_2, v_1) &\leq (\gamma - \alpha) - \frac{1}{2}\lambda(1-\gamma)\delta_1(1 + u_2^2) \\ &= \frac{1}{2}[\{(2\gamma - 2\alpha) - \lambda(1-\gamma)\delta_1\} - \lambda(1-\gamma)\delta_1 u_2^2] \\ &= \frac{1}{2}(A + Bu_2^2),\end{aligned}$$

where

$$\begin{aligned}A &= 2(\gamma - \alpha) - \lambda(1 - \lambda)\delta_1, \\ B &= -\lambda(1-\gamma)\delta_1 \leq 0.\end{aligned}$$

Re $\psi(iu_2, v_1) \leq 0$ if $A \leq 0$ and this gives us value of γ as defined by (3.1). We now apply Lemma 2.1 to conclude that Re $p(z) > 0$ in E and hence $f \in K(\gamma, \delta)$.

Corollary 3.1. *Let $f \in Q^*_\lambda(\alpha, \delta)$, $\lambda \geq 1$. Then $f \in K^*(\gamma_1, \delta)$, where*

$$\gamma_1 = \frac{\alpha(2 + \delta_1) + \delta_1(\lambda - 1)}{2 + \lambda\delta_1},$$

δ_1 is given by (3.2).

Proof. Now

$$\lambda\frac{zf'(z))'}{g'(z)} = \left[(1-\lambda)\frac{zf'(z)}{g(z)} + \lambda\frac{(zf'(z))'}{g'(z)}\right] + (\lambda - 1)\frac{zf'(z)}{g(z)},$$

for $g \in S^*(\delta)$.

Since $f \in Q^*_\lambda(\alpha,\delta)$, we use Theorem 3.1 to have

$$\operatorname{Re}\left\{\lambda\frac{(zf'(z))'}{g'(z)}\right\} > \alpha + (\lambda-1)\left[\frac{2\alpha+\lambda\delta_1}{2+\lambda\delta_1}\right]$$
$$= \frac{\lambda(\alpha\delta_1+2\alpha)+\lambda\delta_1(\lambda-1)}{2+\lambda\delta_1}$$

and this gives us the required result.

Corollary 3.2. *Let λ be complex with $Re\,\lambda \geq 0$, and $f \in Q^*_\lambda(\alpha,1)$. Then, for $z \in E$, $Re\, f'(z) > \frac{2\alpha+Re\lambda}{2+Re\lambda}$.*

This result is proved in [8] with a different method. It can also be proved independently as follows.

Theorem 3.2 *Let $f \in Q^*_\lambda(\alpha,1)$, $0<\alpha<1$, $Re\,\lambda > 0$(λ Complex). Then*

$$Ref'(z) > \sigma_1 = \alpha + (1-\alpha)(2\sigma-1),$$

where σ is given by (2.2). This bound is sharp.

Proof. Let $f'(z) = p(z)$, $p(0)=1$. This implies that

$$f'(z) + \lambda z f''(z) = p(z) + \lambda z p'(z).$$

Since $f \in Q^*_\lambda(\alpha,1)$, Re $\{p(z)+\lambda zp'(z)\} > \alpha$ for $z \in E$. We now apply Lemma 2.3 to obtain the required result.

Corollary 3.3. *Let $\in Q^*_1(0,1)$. Then $Re\, f'(z) > -1+2\log 2 \doteq 0.39$ for $z \in E$. The constant $-1+2\log 2$ cannot be replaced by any larger one as can be seen from the function $f_0 \in Q^*_1(0,1)$ defined by*

$$zf_0'(z) = -z - 2\log(1-z).$$

This result is also proved in [20] by using markedly different techniques.

We now derive the integral representation for the functions in $Q^*_\lambda(\alpha,\delta)$ as follows.

Theorem 3.3 *For $\lambda > 0$, $f \in Q^*_\lambda(\alpha,\delta)$ if and only if there exists $g \in S^*(\delta)$ and a close-to convex function F of order α and type δ such that*

$$zf'(z) = \frac{1}{\lambda}(g(z))^{1-\frac{1}{\lambda}}\int_0^z (g(z))^{\frac{1}{\lambda}-1}F'(t)dt, \qquad (3.4)$$

where all powers are meant as principle values.

Proof. Let $f \in Q^*_\lambda(\alpha,\delta)$. Then from definition 1.1, it follows that

$$(1-\lambda)\frac{zf'(z)}{g(z)} + \lambda\frac{(zf'(z))'}{g'(z)} = p(z), \ \operatorname{Re}p(z) > \alpha \text{ and } g \in S^*(\delta).$$

Multiplying both sides by $[\frac{1}{\lambda}(g(z))^{\frac{1}{\lambda}-1}g'(z)]$, we have

$$(\frac{1}{\lambda}-1)zf'(z)[(g(z))^{\frac{1}{\lambda}-2}g'(z)]+[(zf'(z))'g(z))^{\frac{1}{\lambda}-1}]=\frac{1}{\lambda}p(z)[(g(z))^{\frac{1}{\lambda}-1}(g'(z))]. \quad (3.5)$$

We see that the left-hand side of (3.5) is the exact differential of $zf'(z)(g(z))^{\frac{1}{\lambda}-1}$. Hence, integrating both sides with respect to z and putting $p(z)g(z)=F'(z)$, $F\in K(\alpha,\delta)$, we obtain the required result. The converse follows immediately from (3.4).

Corollary 3.4. *Let $f\in Q^*_\lambda(\alpha,1)$ with $0<\lambda\leq 1$. Then f can be expressed as the Hadamard product of the convex function*

$$k(z)=\frac{1}{\lambda}z^{1-\frac{1}{\lambda}}\int_0^z t^{\frac{1}{\lambda}-2}\frac{t}{1-t}dt, \quad (3.6)$$

with

$$J(z)=\int_0^z p(t)dt,\ Rep(z)>\alpha \quad (3.7)$$

for $z\in E$.

Remark 3.1. Since, for $\lambda\geq 0$, $Q^*_\lambda(\alpha,\delta)\subset K(\gamma,\delta)\subset K$, all functions in $Q^*_\lambda(\alpha,\delta)$, $\lambda\geq 0$ are univalent in E. We notice that $Q^*_0(\alpha,\delta)$ coincides with the class $K(\alpha,\delta)$ and $Q^*_1(\alpha,\delta)\equiv K^*(\alpha,\delta)\subset K(\alpha,\delta)$. Thus we may expect that, as λ increases from 0 to ∞, the classes $Q^*_\lambda(\alpha,\delta)$ decrease and will be nested. We prove this fact as following.

Theorem 3.4. *For $0\leq\lambda_1<\lambda$, $Q^*_\lambda(\alpha,\delta)\subset Q^*_{\lambda_1}(\alpha,\delta)$.*

Proof. For $\lambda_1=0$, the proof is immediate. So we let $\lambda_1>0$ and let $f\in Q^*_\lambda(\alpha,\delta)$. From Definition 1.1 and Theorem 3.1, we can write

$$(1-\lambda_1)\frac{zf'(z)}{g(z)}+\lambda_1\frac{(zf'(z))'}{g'(z)}=\frac{\lambda_1}{\lambda}p(z)+\left(1-\frac{\lambda_1}{\lambda}\right)H(z), \quad (3.8)$$

where $g\in S^*(\delta)$ and

$$\text{Re}p(z)=\text{Re}\left\{(1-\lambda)\frac{zf'(z)}{g(z)}+\lambda\frac{(zf'(z))'}{g'(z)}\right\}>\alpha,$$

$$\text{Re}H(z)=\text{Re}\frac{zf'(z)}{g(z)}>\gamma\geq\alpha.$$

Since the class $P(\alpha)$ consisting of the functions p with $\text{Re}\,p(z)>\alpha$ in E, is a convex set [5], the right hand side of (3.8) belongs to $P(\alpha)$ and this shows that $f\in Q^*_{\lambda_1}(\alpha,\delta)$.

Theorem 3.5. *Let $f\in Q^*_\lambda(\alpha,\delta)$, $\lambda>0$ and be given by (1.1). Then*
(i) $|a_2|\leq\frac{1-\alpha}{1+\lambda}+(1-\delta)$
(ii) $|a_3|\leq\frac{2(1-\alpha)}{3(1+2\lambda)}[1+2(1-\delta)(3-\frac{2}{1+\lambda})]+\frac{(1-\delta)(3-2\delta)}{3}$.

These bounds are sharp as can be seen from the function $f_1 \in Q_\lambda^*(\alpha,\delta)$ and defined by (3.4) with $g(z) = \frac{z}{(1-z)^{2(1-\delta)}}$ and $F(z) = p(z)g(z)$, $p(z) = \frac{1+(1-2\alpha)z}{1-z}$.

Proof. Since $f \in Q_\lambda^*(\alpha,\delta)$, we can write

$$(1-\lambda)zf'(z)g'(z) = p(z)g(z)g'(z),\ g \in S^*(\delta),\ \mathrm{Re}p(z) > \alpha.$$

Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, $P(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$. Then

$$(1-\lambda)[z + \sum_{n=2}^{\infty} na_n z^n][1 + \sum_{n=2}^{\infty} nb_n z^{n-1}] + \lambda[1 + \sum_{n=2}^{\infty} n^2 a_n z^{n-1}]$$
$$= [1 + \sum_{n=1}^{\infty} c_n z^n][1 + \sum_{n=2}^{\infty} nb_n z^n][z + \sum_{n=2}^{\infty} b_n z^n].$$

Thus

$$\begin{aligned} & z + \{(1-\lambda)(2b_2 + 2a_2) + \lambda(b_2 + 4a_2)\}z^2 + \{(1-\lambda)(3b_3 + 4a_2b_2 + 3a_3) \\ & +\lambda(b_3 = 4a_2b_2 + 9a_3)\}z^3 + \cdots \\ &= z + (c_1 + 3b_2)z^2 + (c_2 + 3b_2c_1 + 4b_3 + 2b_2^2)z^3 + \cdots \end{aligned} \tag{3.9}$$

Equating the coefficients of z^2 and z^3 on both sides of (3.9), we have

$$(2-\lambda)b_2 + 2(1+\lambda)a_2 = c_1 + 3b_2,$$

and

$$(3-2\lambda)b_3 + 4a_2b_2 + (3+6\lambda)a_3 = c_2 + 3c_1b_2 + 4b_3 + 2b_2^2.$$

From this it follows that

$$a_2 = \frac{[c_1 + (1+\lambda)b_2]}{2(1+\lambda)}, \tag{3.10}$$

and

$$a_3 = \frac{1}{(3+(1+2\lambda))}\left[c_2 + b_2c_1\left(3 - \frac{2}{1+\lambda}\right)\right] = \frac{b_3}{3}. \tag{3.11}$$

Now, using the known sharp results [4], $|c_n| \le 2(1-\alpha)$ for all $n, |b_2| \le 2(1-\delta)$ and $|b_3| \le \frac{1}{2!}\prod_{m=2}^{3}[1 - 2\delta(m-1)] = (1-\delta)(3-2\delta)$, we have the required estimates from (3.10) and (3.11).

Corollary 3.5 (Covering result). *Let $f \in Q_\lambda^*(a,\delta)$, $\lambda > 0$. If B is the boundary of the image of E under f, then every point of B is at distance at least $\frac{(1+\lambda)}{4+3\lambda-\alpha-\delta(1+\lambda)}$ from the origin.*

Proof. Since f is univalent in E, so is $\frac{cf(z)}{c-f(z)} = z + (a_2 + \frac{1}{c})z^2 + \cdots (c \neq f(z), c \neq 0)$. This implies $|a_2 + \frac{1}{c}| \le 2$ and the result follows on using Theorem 3.5.

Theorem 3.6. *Let $f \in Q_0^*(\alpha,\delta)$, then $f \in Q_1^*(\alpha,\delta)$ for $|z| < r_0$ where*

$$r_0 = \begin{cases} \frac{(1-2\delta)-\sqrt{\delta^2-2\delta+3}}{(1-2\delta)}, \delta \neq \frac{1}{2} \\ \frac{1}{3}, \delta = \frac{1}{2}. \end{cases} \tag{3.12}$$

Proof. We can write

$$zf'(z) = g(z)h(z),\ g \in S^*(\delta),\ \mathrm{Re}h(z) > \alpha.$$

Simple manipulations give us

$$\mathrm{Re}\frac{(zf'(z))'}{g'(z)} - \alpha > \mathrm{Re}h(z) - \alpha - |\frac{g(z)}{g'(z)}||h'(z)|.$$

Using the well-known results, see [5],

$$|\frac{g(z)}{g'(z)}| \leq \frac{r(1+r)}{1-(1-2\delta)r},$$

and

$$|h'(z)| \leq \frac{2[\mathrm{Re}h(z)-\alpha]}{1-r^2},$$

we have

$$\mathrm{Re}\left\{\frac{(zf'(z))'}{g'(z)}\right\} - \alpha \geq [\mathrm{Re}h(z)-\alpha]\left\{\frac{1-(4-2\delta)r+(1-2\delta)r^2}{(1-r)(1-(1-2\delta)r)}\right\}. \tag{3.13}$$

The righthand side of (3.13) is positive for $|z| < r_0$, where r_0 is given by (3.12).

We now consider the converse case of Theorem 3.6 as follows.

Theorem 3.7. *Let $f \in Q_1^*(\alpha,\delta)$. Then $f \in Q_0^*(\beta,\delta)$ and β is given by*

$$\beta = \beta(\alpha) = \begin{cases} \frac{2\alpha-1}{2(1-2^{1-2\alpha})}, & \alpha \neq \frac{1}{2} \\ \frac{1}{2\log 2}, & \alpha = \frac{1}{2} \end{cases} \tag{3.14}$$

Proof. Since $f \in Q_1^*(\alpha,\delta)$, we can write

$$\frac{(zf'(z))'}{g'(z)} = (1-\alpha)h(z)+\alpha,\ \mathrm{Re}h(z) > 0,\ g \in S^*(\delta)$$
$$= (1-\alpha)\frac{zs'(z)}{s(z)} + \alpha,\ \text{for some } s \in S^*.$$

Thus we can write

$$\frac{(zf'(z))'}{g'(z)} = \frac{[z(\frac{s(z)}{z})^{1-\alpha}]'}{(\frac{s(z)}{z})^{1-\alpha}} = \frac{N'}{D'}\ \text{(say)}$$

Hence

$$\frac{N}{D} = \frac{zf'(z)}{g(z)} = \frac{z(\frac{s(z)}{z})^{1-\alpha}}{\int_0^z (\frac{s(t)}{t})^{1-\alpha} dt}.$$

Now let

$$G(z) = \int_0^z (\frac{s(t)}{t})^{1-\alpha} dt.$$

It is known [3] that, if $s \in S^*$, $G \in S^*(\beta)$ where β is given by (3.14). This completes the proof.

4. The Class $Q_\lambda^*(\alpha, 1)$

In this section we shall discuss the class $Q_\lambda^*(\alpha, 1)$ in more detail.

Theorem 4.1. *The class $Q_\lambda^*(\alpha, 1)$ is a convex set.*

Proof. Let $f, g \in Q_\lambda^*(\alpha, 1)$ and let, for $0 \leq \beta_1 \leq 1$,

$$F(z) = \beta_1 f(z) + (1 - \beta_1)g(z).$$

Then

$$\begin{aligned} F'(z) + \lambda z F''(z) &= [\beta_1 f'(z) + (1-\beta_1)g'(z)] + \lambda z[\beta_1 f''(z) + (1-\beta_1)g''(z)] \\ &= \beta_1[f'(z) + \lambda z f''(z)] + (1-\beta_1)[g'(z) + \lambda z g''(z)] \\ &= \beta_1 p_1(z) + (1-\beta_1)p_2(z) = p(z). \end{aligned}$$

Since $p_1, p_2 \in P(\alpha)$ and $P(\alpha)$ is a convex set, $p \in P(\alpha)$. Hence $F \in Q_\lambda^*(\alpha, 1)$.

We now show that the class $Q_\lambda^*(\alpha, 1)$ is closed under convolution with convex functions.

Theorem 4.2. *Let $\phi \in C$ and $f \in Q_\lambda^*(\alpha, 1)$. Then $\phi * f \in Q_\lambda^*(\alpha, 1)$.*

Proof. Let $H = \phi * f$. Then

$$\begin{aligned} H'(z) + \lambda z H''(z) &= (\phi * f)'(z) + \lambda z(\phi * f)''(z) \\ &= [\frac{\phi(z)}{z} * f'(z)] + [\frac{\phi(z)}{z} * zf''(z)] \\ &= \frac{\phi(z)}{z} * [f'(z) + \lambda z f''(z)]. \end{aligned}$$

Since ϕ in convex, we have Re $\frac{\phi(z)}{z} > \frac{1}{2}$ for $z \in E$, [21]. Thus, using Lemma 2.4 and the given fact that $f \in Q_\lambda^*(\alpha, 1)$, we obtain the required result.

In fact, we have the following more general result.

Theorem 4.3. *Let g analytic in E, $g(0) = 0$, $g'(0) = 1$ and satisfy the condition $Re\frac{g(z)}{z} > \frac{1}{2}$, $z \in E$. Let $f \in Q_\lambda^*(\alpha, 1)$. Then $f * g \in Q_\lambda^*(\alpha, 1)$.*

Corollary 4.1. *Let $f \in Q_\lambda^*(\alpha, 1)$. Then $Q_\lambda^*(\alpha, 1)$ is invariant under the following integral operators.*

(i) $f_1(z) = \int_0^z \frac{f(t)}{t} dt$

(ii) $f_2(z) = \frac{2}{z}\int_0^z f(t)dt$ (Libera's opetator [11])

(iii) $f_3(z) = \int_0^z \frac{f(t)-f(xt)}{t-xt} dt$, $|x| \leq 1$, $x \neq 1$

(iv) $f_4(z) = \frac{1+c}{z^c}\int_0^z \xi^{c-1} f(t)dt$, Rec > 0.

Proof. we may write, see [1],

$$f_i(z) = (f * \phi_i)(z), \; i = 1, 2, 3, 4$$

where ϕ_i are convex for all i and

$$\phi_1(z) = -\log(1-z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n,$$

$$\phi_2(z) = \frac{-2[z + \log(1-z)]}{z} = \sum_{n=1}^{\infty} \frac{2}{n+1} z^n,$$

$$\phi_3(z) = \frac{1}{1-x} \log[\frac{1-xz}{1-z}] = \sum_{n=1}^{\infty} \frac{(1-x^n)}{(1-x)n} z^n, \; |x| \leq 1, x \neq 1,$$

$$\phi_4(z) = \sum_{n=1}^{\infty} \frac{1+c}{n+c} z^n, \text{Rec} \geq 0, \text{ see[18]}.$$

Now the result follows by applying Theorem 4.2.

Let u_1 and u_2 be linear operators defined as follows.

$$u_1(f(z)) = zf'(z),$$
$$u_2(f(z)) = [f(z) + zf'(z)]/2. \text{ (Livingston's operator[12])}$$

Both these operators can be written as a convolution operator [1] given by

$$u_i(f) = h_i * F, \quad i = 1, 2$$

where

$$h_1(z) = \sum_{n=1}^{\infty} = \frac{z}{(1-z)^2},$$

$$h_2(z) = \sum_{n=1}^{\infty} \frac{n+1}{2} z^n = \frac{z - z^2/2}{(1-z)^2}.$$

It can easily be verified that the radius of convexity $r_c(h_1) = 2 - \sqrt{3}$ and $r_c(h_2) = 1/2$. These facts together with Theorem 4.2 yield the following.

Corollary 4.2. *Let $f \in Q^*_\lambda(\alpha, 1)$. Then $u_1(f) = f * h_1 \in Q^*_\lambda(\alpha, 1)$ for $|z| < 2 - \sqrt{3}$ and $u_2(f) = f * h_2 \in Q^*_\lambda(\alpha, 1)$ for $|z| < 1/2$.*

Next we find the radius of convexity for $f \in Q^*_\lambda(\alpha, 1)$ under certain conditions.

Theorem 4.4. *Let $f \in Q^*_\lambda(\alpha, 1)$, $0 < \lambda \le 1$, $0 < \alpha < 1/3$. Then f maps $|z| < R$ onto a convex domain where $R = min(r_1, r_2)$, r_1 is the unique root of the equation (2.1) in the interval (0,1] and r_2 is given by*

$$r_2 = 1/[(1 - 2\alpha) + \sqrt{2(1 - \alpha)(1 - 2\alpha)}]. \tag{4.1}$$

This result is sharp.

Proof. $f \in Q^*_\lambda(\alpha, 1)$, $0 < \lambda \le 1$ implies that

$$f(z) = (k * J)(z),$$

where k is defined by (3.6) and J is given by (3.7). If we show that $J \in C$ for $|z| < R$, then $f = k * J$ is in C for $|z| < R$, see [19].

Now, from (3.7) and Lemma 2.2, we have

$$\mathrm{Re}\Big(1 + \frac{zJ''(z)}{J'(z)}\Big) = 1 + \mathrm{Re}\frac{zp'(z)}{p(z)} \ge 1 - \frac{2(1-\alpha)r}{[1-(1-2\alpha)r](1+r)}, \qquad (r < r_1)$$
$$= \frac{1 - 2(1-2\alpha)r - (1-2\alpha)r^2}{[1-(1-2\alpha)r](1+r)}, \qquad \text{for } r < r_1,$$

where r_1 is the unique root of (2.1) in (0,1]. Let $T(r) = 1 - 2(1 - 2\alpha)r - (1 - 2\alpha)r^2$. Then $T(0) = 1 > 0$ and $T(1) = -2 + 6\alpha < 0$ for $\alpha < 1/3$. Therefore $T(r)$ has at least one root in (0,1]. Let $r_2 < 1$ be the positive smaller root of $T(r) = 0$. Then r_2 is given by (4.1). Hence $J \in C$ for $|z| < R$.

Sharpness follows from the function

$$f(z) = k(z) * \int_0^z p_e(t)dt,$$

where

$$p_e(z) = \frac{1 + (1 - 2\alpha)z}{1 - z}.$$

We note that $R = \sqrt{2} - 1$ for $\alpha = 0$.

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