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GENERALIZATIONS OF COPSON'S INEQUALITITS INVOLVING SERIES OF POSITIVE TERMS

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Abstract. The aim of the present paper is to establish some new inequalities involving series of positive terms.

1. Introduction

In [1] Copson established the following Hardy's inequalities [4, Theorem 326 and Theorem 331] involving series of positive terms.

Theorem A. *If* $p > 1$, $\lambda_n > 0$, $a_n > 0$, $\Lambda_n = \sum_{i=1}^{n} \lambda_i$, $A_n = \sum_{i=1}^{n} \lambda_i a_i$ *and* $\sum_{n=1}^{\infty} \lambda_n a_n^p$ *converges, then*

$$\sum_{n=1}^{\infty} \lambda_n (\frac{A_n}{\Lambda_n})^p \leq (\frac{p}{p-1})^p \sum_{n=1}^{\infty} \lambda_n a_n^p. \tag{1}$$

The constant is the best possible.

Theorem B. *Let* $p, \lambda_n, a_n, \Lambda_n$ *be defined as in Theorem A. If* $A_n = \sum_{i=n}^{\infty} \frac{\lambda_i a_i}{\Lambda_i}$ *and* $\sum_{n=1}^{\infty} \lambda_n a_n^p$ *converges, then*

$$\sum_{n=1}^{\infty} \lambda_n A_n^p \leq p^p \sum_{n=1}^{\infty} \lambda_n a_n^p. \tag{2}$$

The constant is the best possible.

There is a vast literature which deals with alternative proofs, generalizations and extensions of (1) and (2), see [2,3,5,6,8,12] and the references given therein. In the present paper we establish some new inequalities involving series of positive terms which claim their origin to the inequalities given in (1) and (2).

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2. Main results

The following Theorems are base on the idea used by Levinson [7] to obtain the interesting generalizations of Hardy's integral inequality and Pachpatte and Love [11] to obtain inequalities related to Hardy's integral inequality. In this section, we establish some discrete analogue of theirs. Here we assume that the left sides of inequalities exist when right sides do.

Theorem 1. *For $n = 1, 2, 3, \ldots$ and $m = 1, 2, 3, \ldots, M$, let $p > 1$, $a(n) > 0$, $\lambda_m(n) > 0$, $\beta_m(n) > 0$, $\Lambda_m(n) = \sum_{i=1}^{n} \lambda_m(i)\beta_m(i)$, $I_m a(n) = \frac{1}{\Lambda_m(n)} \sum_{i=1}^{n} \lambda_m(i)\beta_m(i)a(i)$, $A_m(n) = I_m I_{m-1} \cdots I_1 a(n)$, $A_0(n) = a(n)$, where M is a positive integer, and further let $\sum_{n=1}^{\infty} \lambda_1(n)\, A_0^p(n)$ converge. If there exists $k_m \geq p/(p-1)$ such that*

$$p - 1 + \frac{[\beta_m(n+1) - \beta_m(n)]\Lambda_m(n)}{\beta_m(n+1)\beta_m(n)\lambda_m(n)} \geq \frac{p}{k_m}, \text{ and } \lambda_1(n) \geq \lambda_2(n) \geq \cdots \geq \lambda_M(n), \tag{3}$$

then

$$\sum_{n=1}^{\infty} \lambda_M(n) A_M^p(n) \leq (\pi_{m=1}^{M} k_m)^p \sum_{n=1}^{\infty} \lambda_1(n) A_0^p(n), \tag{4}$$

the constant in (4) is the best possible.

Proof. Let $\lambda_m(0) = \beta_m(0) = 1$ and agree that $\Lambda_m(0) = 0$ for $m = 1, 2, 3, \ldots, M$. For $n = 0, 1, 2, \ldots$ and $m = 1, 2, 3, \ldots, M$,

$$\begin{aligned}
&-p\lambda_m(n+1)A_{m-1}(n+1)A_m^{p-1}(n+1) \\
&= -p\lambda_m(n+1)\beta_m(n+1)A_{m-1}(n+1)[A_m^{p-1}(n+1)/\beta_m(n+1)] \\
&= -p[A_m(n+1)\Lambda_m(n+1) - A_m(n)\Lambda_m(n)][A_m^{p-1}(n+1)/\beta_m(n+1)] \\
&= -pA_m^p(n+1)[\Lambda_m(n+1)/\beta_m(n+1)] + p[\Lambda_m(n)/\beta_m(n+1)]A_m(n)A_m^{p-1}(n+1) \\
&\leq -pA_m^p(n+1)[\Lambda_m(n+1)/\beta_m(n+1)] + [\Lambda_m(n)/\beta_m(n+1)] \cdot A_m^p(n) \\
&\quad +(p-1)[\Lambda_m(n)/\beta_m(n+1)]]A_m^p(n+1),
\end{aligned}$$

the last inequality follows from Young's inequality i.e.
$kxy^{k-1} \leq x^k + (k-1)y^k, x, y \geq 0,\ k > 1$.

Hence

$$\begin{aligned}
&(p-1)\lambda_m(n+1)A_m^p(n+1) + \frac{[\beta_m(n+1) - \beta_m(n)]\Lambda_m(n)}{\beta_m(n+1)\beta_m(n)\lambda_m(n)}\lambda_m(n)A_m^p(n) \\
&-p\lambda_m(n+1)A_{m-1}(n+1)A_m^{p-1}(n+1) \\
&\leq (p-1)\lambda_m(n+1)A_m^p(n+1) + \frac{[\beta_m(n+1) - \beta_m(n)]\Lambda_m(n)}{\beta_m(n+1)\beta_m(n)}A_m^p(n) \\
&-pA_m^p(n+1)[\Lambda_m(n+1)/\beta_m(n+1)] \\
&+[\Lambda_m(n)/\beta_m(n+1)]A_m^p(n) + (p-1)[\Lambda_m(n)/\beta(n+1)]A_m^p(n+1) \\
&= \frac{\Lambda_m(n)A_m^p(n)}{\beta_m(n)} - \frac{\Lambda_m(n+1)A_m^p(n+1)}{\beta_m(n+1)}
\end{aligned}$$

By adding the inequalities for $n = 0, 1, 2, \ldots, N-1$, we have

$$\sum_{n=0}^{N-1}(p-1)\lambda_m(n+1)A_m^p(n+1) + \sum_{n=0}^{N-1}\frac{[\beta_m(n+1)-\beta_m(n)]\Lambda_m(n)}{\beta_m(n+1)\beta_m(n)\lambda_m(n)}\lambda_m(n)A_m^p(n)$$
$$-p\sum_{n=0}^{N-1}\lambda_m(n+1)A_{m-1}(n+1)A_m^{p-1}(n+1)$$
$$\leq -[\Lambda(N)/\beta_m(N)]A_m^p(N) \leq 0.$$

Thus,

$$\sum_{n=1}^{N}(p-1)\lambda_m(n)A_m^p(n) + \sum_{n=1}^{N-1}\frac{[\beta_m(n+1)-\beta_m(n)]\Lambda_m(n)}{\beta_m(n+1)\beta_m(n)\lambda_m(n)}\lambda_m(n)A_m^p(n)$$
$$\leq p\sum_{n=1}^{N}\lambda_m(n)A_{m-1}(n)A_m^{p-1}(n).$$

Using (3) and the assumption that $k_m \geq p/(p-1)$, we have

$$\sum_{n=1}^{N}\lambda_m(n)A_m^p(n) \leq K_m\sum_{n=1}^{N}\lambda_m(n)A_{m-1}(n)A_m^{p-1}(n). \tag{5}$$

Using Hölder inequality with indices p and $p/(p-1)$, we have

$$\sum_{n=1}^{N}\lambda_m(n)A_{m-1}(n)A_m^{p-1}(n) = \sum_{n=1}^{N}\lambda_m^{1/p}(n)A_{m-1}(n)\lambda_m^{(p-1)/p}(n)A_m^{p-1}(n)$$
$$\leq [\sum_{n=1}^{N}\lambda_m(n)A_{m-1}^p(n)]^{1/p}[\sum_{n=1}^{N}\lambda_m(n)A_m^p(n)]^{(p-1)/p},$$

which together with (5), imply

$$\sum_{n=1}^{N}\lambda_m(n)A_m^p(n) \leq k_m\Big[\sum_{n=1}^{N}\lambda_m(n)A_{m-1}^p(n)\Big]^{1/p}\Big[\sum_{n=1}^{N}\lambda_m(n)A_m^p(n)\Big]^{(p-1)/p}.$$

Dividing the above inequality by the last factor on the right side and raising the result to the pth power, we obtain

$$\sum_{n=1}^{N}\lambda_m(n)A_m^p(n) \leq k_m^p\sum_{n=1}^{N}\lambda_m(n)A_{m-1}^p(n).$$

Now since $\lambda_1(n) \geq \lambda_2(n) \geq \cdots \geq \lambda_M(n)$ for $n = 1, 2, 3, \ldots$, we have

$$\sum_{n=1}^{N}\lambda_M(n)A_M^P(n) \leq (\pi_{m=1}^M K_m)^p\sum_{n=1}^{N}\lambda_1(n)A_0^p(n). \tag{6}$$

The desired inequality (4) then follows from (6) by letting N tend to infinity.

The case $M = 1$, $\beta_1(n) = 1$, $n = 1, 2, 3, \ldots, k_1 = p/(p-1)$ show constant in (4) is the best possible.

Remark 1. Theorem 1 redues to Theorem 1 in [5] when $M = 1$ and reduces to (1) when $M = 1$, $K_1 = p/(p-1)$ and $\beta_1(n) = 1$ for $n = 1, 2, 3, \ldots$.

Theorem 2. *Let H be a real-valued positive convex function defined on $(0, \infty)$, and let p, $a(n)$, $\lambda_m(n)$, $\beta_m(n)$, $\Lambda_m(n)$, $I_m a(n)$, $A_m(n)$, $A_0(n)$ and k_m be as in Theorem 1. If $\sum_{n=1}^{\infty} \lambda_1(n) H^p(A_0(n))$ converges, then*

$$\sum_{n=1}^{\infty} \lambda_M(n) H^p(A_M(n)) \le (\pi_{m=1}^{M} k_m)^p \sum_{n=1}^{\infty} \lambda_1(n) H^p(A_0(n)). \tag{7}$$

Proof. Since H is a convex function, by repeated application of Jensen's inequality, we obtain

$$H(A_M(n)) \le F(n), \quad \text{where } F(n) = I_M I_{M-1} \cdots I_1 H(a(n)).$$

Thus

$$\sum_{n=1}^{\infty} \lambda_M(n) H^p(A_M(n)) \le \sum_{n=1}^{\infty} \lambda_M(n) F^p(n). \tag{8}$$

Replace $a(n)$ by $H(a(n))$ is (4), we have

$$\sum_{n=1}^{\infty} \lambda_M(n) F^p(n) \le (\pi_{m=1}^{M} k_m)^p \sum_{n=1}^{\infty} \lambda_1(n) H^p(a(n)). \tag{9}$$

The inequality (7) then follows from (8) and (9).

Remark 2. The inequality (4) is the special case of the inequality (7) when $H(u) = u$. Theorem 2 redues to Theorem 2 in [5] when $M = 1$ and reduced to Theorem 1 in [12] when $M = 1$, $k_1 = p/(p-1)$ and $\beta_1(n) = 1$, for $n = 1, 2, 3, \ldots$. We note that the last case shows the constant in (7) is the best possible.

Theorem 3. *Let p, $a(n)$, $\lambda_m(n)$, $\beta_m(n)$, $\Lambda_m(n)$, $I_m a(n)$, $A_m(n)$, $A_0(n)$ and k_m be as in Theorem 1, and let $\varphi > 0$ be defined on $(0, \infty)$ so that $\varphi'' \ge 0$ and*

$$\varphi\varphi'' \ge (1 - 1/p)(\varphi')^2. \tag{10}$$

If $\sum_{n=1}^{\infty} \lambda_1(n) \varphi(A_0(n))$ converges, then

$$\sum_{n=1}^{\infty} \lambda_M(n) \varphi(A_M(n)) \le (\pi_{m=1}^{M} k_m)^p \sum_{n=1}^{\infty} \lambda_1(n) \varphi(A_0(n)) \tag{11}$$

Proof. Let $\Psi(u) = \varphi^{1/p}(u), u > 0$. Then, by (10), $\psi'' \geq 0$. Hence ψ is convex on $(0, \infty)$. Thus, by Theorem 2, we have

$$\sum_{n=1}^{\infty} \lambda_M(n)\psi^p(A_M(n)) \leq (\pi_{m=1}^M k_m)^p \sum_{n=1}^{\infty} \lambda_1(n)\psi^p(A_0(n)),$$

and therefore

$$\sum_{n=1}^{\infty} \lambda_M \varphi(A_M(n)) \leq (\pi_{m=1}^M k_m)^p \sum_{n=1}^{\infty} \lambda_1(n)\varphi(A_0(n)).$$

This is the desired inequality (11).

Remark 3. Theorem 3 reduces to Theorem 2 and Theorem 1 when $\varphi(u) = H^p(u)$ and $\varphi(u) = u^p$, respectively. Theorem 3 reduces to Theorem 3 in [5] when $M = 1$, and reduces to Theorem 1 in [12] when $M = 1$, $\varphi(u) = H^p(u)$, $k_1 = p/(p-1)$ and $\beta_1(n) = 1$, for $n = 1, 2, \ldots$. Also we note that the inequality (1) is the special case of the inequality (11) when $M = 1$, $\varphi(u) = u^p$, $k_1 = p/(p-1)$ and $\beta_1(n) = 1$, for $n = 1, 2, \ldots$.

Theorem 4. *For $n = 1, 2, 3, \ldots$ and $m = 1, 2, \ldots, M$ let $p > 1$, $a(n) > 0$, $\lambda_m(n-1) > 0$, $\langle\beta_m(n-1)\rangle$ be non-increasing positive sequence, $\Lambda_m(n) = \sum_{i=1}^n \lambda_m(i)\beta_m(i)$, $\Lambda_m(0) = 0$, $J_m a(n) = \sum_{i=n}^{\infty} \frac{\lambda_m(i)\beta_m(i)a(i)}{\Lambda_m(i)}$, $B_m(n) = J_m J_{m-1} \cdots J_1 a(n)$, $B_0(n) = a(n)$ and futher let $\sum_{n=1}^{\infty} \lambda_m(n)B_{m-1}^p(n)$ converge for each m. If there exist $k_m \geq p$ such that*

$$1 - \frac{[\beta_m(n) - \beta_m(n-1)]\Lambda_m(n-1)}{\beta_m(n)\beta_m(n-1)\lambda_m(n)} \geq \frac{p}{k_m} \tag{12}$$

and further if $\lambda_1(n) \geq \lambda_2(n) \geq \cdots \geq \lambda_M(n)$ then

$$\sum_{n=1}^{\infty} \lambda_M(n)B_M^p(n) \leq (\pi_{m=1}^M k_m)^p \sum_{n=1}^{\infty} \lambda_1(n)B_0^p(n). \tag{13}$$

Proof. For $n = 1, 2, \ldots$ and $m = 1, 2, \ldots, M$, we have, by Hölder's inequality,

$$\begin{aligned} B_m^p(n) &= [\sum_{i=n}^{\infty} \lambda_m^{1/p}(i)B_{m-1}(i)\lambda_m^{(p-1)/p}(i)\beta_m(i)\Lambda_m^{-1}(i)]^p \\ &\leq \Big[\sum_{i=n}^{\infty} \lambda_m(i)B_{m-1}^p(i)\Big]\Big[\sum_{i=n}^{\infty} \lambda_m(i)\Big(\frac{\Lambda_m(i)}{\beta_m(i)}\Big)^{-p/(p-1)}\Big]^{p-1}. \end{aligned}$$

Since

$$\begin{aligned} \sum_{i=n}^{\infty} \lambda_m(i)\Big(\frac{\Lambda_m(i)}{\beta_m(i)}\Big)^{-p/(p-1)} &\leq \sum_{i=n}^{\infty} \Big[\lambda_m(i) + \frac{(\beta_m(i-1) - \beta_m(i))\Lambda_m(i-1)}{\beta_m(i-1)\beta_m(i)}\Big] \\ &\quad \cdot \Big(\frac{\Lambda_m(i)}{\beta_m(i)}\Big)^{-p/(p-1)} \end{aligned}$$

$$< \int_{\frac{\Lambda_m(n-1)}{\beta_m(n-1)}}^{\infty} x^{-p/(p-1)}dx$$
$$= (p-1)\left(\frac{\Lambda_m(n-1)}{\beta_m(n-1)}\right)^{-1/(p-1)}.$$

Hence, we have

$$B_m^p(n) < (p-1)^{p-1}R_m(n-1)\left(\frac{\Lambda_m(n-1)}{\beta_m(n-1)}\right)^{-1}, \tag{14}$$

where $R_m(n-1) = \sum_{i=n}^{\infty} \lambda_m(i)B_{m-1}^p(i)$.
Also

$$\begin{aligned}
&-p\lambda_m(n)B_{m-1}(n)B_m^{p-1}(n)\\
&= -p\lambda_m(n)\beta_m(n)B_{m-1}(n)\frac{B_m^{p-1}(n)}{\beta_m(n)}\\
&= -p\Lambda_m(n)(B_m(n) - B_m(n+1))\left(\frac{B_m^{p-1}(n)}{\beta_m(n)}\right)\\
&= -p\left(\frac{\Lambda_m(n)}{\beta_m(n)}\right)B_m^p(n) + P\left(\frac{\Lambda_m(n)}{\beta_m(n)}\right)B_m(n+1)B_m^{p-1}(n)\\
&\leq -p\left(\frac{\Lambda_m(n)}{\beta_m(n)}\right)B_m^p(n) + \left[\frac{\Lambda_m(n)}{\beta_m(n)}\right][B_m^p(n+1) + (p-1)B_m^p(n)]\\
&= \left(\frac{\Lambda_m(n)}{\beta_m(n)}\right)[B_m^p(n+1) - B_m^p(n)].
\end{aligned}$$

Thus

$$\begin{aligned}
&\lambda_m(n)B_m^p(n) - \frac{(\beta_m(n) - \beta_m(n-1)\Lambda_m(n-1)}{\beta_m(n)\beta_m(n-1)\lambda_m(n)}\lambda_m(n)B_m^p(n)\\
&\quad -p\lambda_m(n)B_{m-1}(n)B_m^{p-1}(n)\\
&\leq \lambda_m(n)B_m^p(n) - \left(\frac{\Lambda_m(n-1)}{\beta_m(n-1)}\right)B_m^p(n) + \left(\frac{\Lambda_m(n-1)}{\beta_m(n)}\right)B_m^p(n)\\
&\quad + \left(\frac{\Lambda_m(n)}{\beta_m(n)}\right)B_m^p(n+1) - \left(\frac{\Lambda_m(n)}{\beta_m(n)}\right)B_m^p(n)\\
&= \left(\frac{\Lambda_m(n)}{\beta_m(n)}\right)B_m^p(n+1) - \left(\frac{\Lambda_m(n-1)}{\beta_m(n-1)}\right)B_m^p(n)
\end{aligned}$$

for $n = 1, 2, 3, \ldots,$.

By adding the above inequalities for $n = 1, 2, 3, \ldots, N$ and using (14), we have

$$\sum_{n=1}^{N} \lambda_m(n)B_m^p(n) - \sum_{n=1}^{N} \frac{(\beta_m(n) - \beta_m(n-1))\Lambda_m(n-1)}{\beta_m(n)\beta_m(n-1)\lambda_m(n)}\lambda_m(n)B_m^p(n)$$

$$
\begin{aligned}
&-p\sum_{n=1}^{N}\lambda_m(n)B_{m-1}(n)B_m^{p-1}(n)\\
&\leq \left(\frac{\Lambda_m(N)}{\beta_m(N)}\right)B_m^p(N+1)\\
&< (p-1)^{p-1}R_m(N).
\end{aligned}
$$

Using (12) and the assumption that $k_m \leq p$, we obtain

$$\sum_{n=1}^{N}\lambda_m(n)B_m^p(n) < K_m\Big(\sum_{n=1}^{N}\lambda_m(n)B_{m-1}(n))B_m^{p-1}(n)+\varepsilon_m(N)\Big) \tag{15}$$

where $\varepsilon_m(N)=\frac{1}{p}(p-1)^{p-1}R_m(N)$ tend to zero when $N\to\infty$.

Applying Hölder's inequality to the right hand side of (15) gives

$$\Big(\sum_{n=1}^{N}\lambda_m(n)B_{m-1}(n)B_m^{p-1}(n)\Big) \leq \Big(\sum_{n=1}^{N}\lambda_m(n)B_{m-1}^p(n)\Big)^{1/p}\Big(\sum_{n=1}^{N}\lambda_m(n)B_m^p(n)\Big)^{(p-1)/p},$$

so that

$$\sum_{n=1}^{N}\lambda_m(n)B_m^p(n) < k_m\left[\Big(\sum_{n=1}^{N}\lambda_m(n)B_{m-1}^p(n)\Big)^{1/p}\Big(\sum_{n=1}^{N}\lambda_m(n)B_m^p(n)\Big)^{(p-1)/p}+\varepsilon_m(N)\right]. \tag{16}$$

By (16), we have

$$
\begin{aligned}
\Big(\sum_{n=1}^{N}\lambda_m(n)B_m^p(n)\Big)^{1/p} < k_m\Bigg[&\Big(\sum_{n=1}^{N}\lambda_m(n)B_{m-1}^p(n)\Big)^{1/p}\\
&+\varepsilon_m(N)/\Big(\sum_{n=1}^{N}\lambda_m(n)B_m^p(n)\Big)^{(p-1)/p}\Bigg].
\end{aligned} \tag{17}
$$

By letting N tend to infinity in (17) and raising the result to the pth power, we have

$$\sum_{n=1}^{\infty}\lambda_m(n)B_m^p(n) \leq k_m^p\sum_{n=1}^{\infty}\lambda_m(n)B_{m-1}^p(n) \tag{18}$$

for $m=1,2,\ldots,M$. Now, since $\lambda_1(n)\geq\lambda_2(n)\geq\cdots\geq\lambda_M(n)$ for $n=1,2,3,\ldots$, we have

$$\sum_{n=1}^{\infty}\lambda_M(n)B_M^p(n) \leq \left(\pi_{m=1}^{M}k_m\right)^p\sum_{n=1}^{\infty}\lambda_1(n)B_0^p(n).$$

This is the desired inequality (13).

Remark 4. We note that Theorem 4 is a discrete analogue of Theorem 4 in [7] when $M=1$. Also we note that the inequality (2) is the special case of the inequality (13) when $M = 1$, $k_1 = p$ and $\beta_1(n) = 1$, for $n = 1, 2, 3, \ldots$, which shows the constant in (13) is the best possible.

The following Theorems are discrete analogue results given by Pachpatte in [10] which claim thier origin in the Copson's inequality givin in (1) and (2).

Theorem 5. *Let H be defined as in Theorem 2 and for $n = 1, 2, 3, \ldots,$ and $j = 1, 2.,$ let*

$$p_j > 1, 1/p_1 + 1/p_2 = 1, \lambda(n) > 0, \beta_j(n) > 0, a_j(n) > 0,$$
$$\Lambda_j(n) = \sum_{i=1}^{n} \lambda(i)\beta_j(i), A_j(n) = \sum_{i=1}^{n} \lambda(i)\beta_j(i)a_j(i) \text{ and } \sum_{n=1}^{\infty} \lambda(n)H^{p_j}(a_j(n))$$

converges. If there exists $U_j \geq p_j/(p_j - 1)$ such that

$$p_j - 1 + \frac{(\beta_j(n+1) - \beta_j(n)) - \Lambda_j(n)}{\beta_j(n+1)\beta_j(n)\lambda(n)} \geq \frac{p_j}{U_j}, \quad \text{then}$$

$$\sum_{n=1}^{\infty} \lambda(n)H\Big(\frac{A_1(n)}{\Lambda_1(n)}\Big)H\Big(\frac{A_2(n)}{\Lambda_2(n)}\Big) \leq \frac{U_1^{p_1}}{p_1}\sum_{n=1}^{\infty} \lambda(n)H^{p_1}(a_1(n)) + \frac{U_2^{p_2}}{p_2}\sum_{n=1}^{\infty} \lambda(n)H^{p_2}(a_2(n)). \tag{19}$$

Proof. By the elementary inequality (see [9, p.30])

$$xy \leq \frac{1}{p_1}x^{p_1} + \frac{1}{p_2}y^{p_2}$$

where $x, y \geq 0$, $p_1 > 1$ and $1/p_1 + 1/p_2 = 1$, we observe that

$$\sum_{n=1}^{\infty} \lambda(n)H\Big(\frac{A_1(n)}{\Lambda_1(n)}\Big)H\Big(\frac{A_2(n)}{\Lambda_2(n)}\Big) \leq \frac{1}{p_1}\sum_{n=1}^{\infty} \lambda(n)H^{p_1}\Big(\frac{A_1(n)}{\Lambda_1(n)}\Big) + \frac{1}{p_2}\sum_{n=1}^{\infty} \lambda(n)H^{p_2}\Big(\frac{A_2(n)}{\Lambda_2(n)}\Big).$$

Now a suitable application of Theorem 2 when $M = 1$ on the right side of the above inequality yields the required inequality in (19) and the proof of Theorem 5 is complete.

Theorem 6. *For $n = 1, 2, 3, \cdots,$ and $j = 1, 2,$ let $p_j, a_j(n), \Lambda_j(n)$ be defined as in theorem 5 and let $\lambda(n-1), \Lambda_j(0) = 0, < \beta_j(n-1) >,$ be non-increasing positive sequence and $B_j(n) = \sum_{i=n}^{\infty} \frac{\lambda(i)B_j(i)a_j(i)}{\Lambda_j(i)}$, and further let $\sum_{n=1}^{\infty} \lambda(n)a_j^{p_j}(n)$ converge. If there exists $V_j \geq P_j$ such that*

$$1 - \frac{(\beta_j(n) - \beta_j(n-1))\Lambda_j(n-1)}{\beta_j(n)\beta_j(n-1)\lambda(n)} \geq \frac{P_j}{V_j},$$

then

$$\sum_{n=1}^{\infty} \lambda(n)B_1(n)B_2(n) \leq \frac{V_1^{p_1}}{p_1}\sum_{n=1}^{\infty} \lambda(n)a_1^{p_1}(n) + \frac{V_2^{p_2}}{p_2}\sum_{n=1}^{\infty} \lambda(n)a_2^{p_2}(n).$$

Proof. By the same steps as in the proof of Theorem 5 with suitable modifications and replacing Theorem 2 by Theorem 4.

Remark 5. If we take $H(u) = u$, $p_1 = p_2 = 2$, $a_j(n) = a(n)$, $\Lambda_j(n) = \Lambda(n)$ and $A_j(n) = A(n)$; $B_j(n) = B(n)$ in Theorem 5 and Theorem 6, then Theorem 5 and Theorem 6 reduce respectively to

$$\sum_{n=1}^{\infty}\lambda(n)\Big(\frac{A(n)}{\Lambda(n)}\Big)^2 \le \Big(\frac{U_1^2+U_2^2}{2}\Big)\sum_{n=1}^{\infty}\lambda(n)a^2(n) \tag{20}$$

and

$$\sum_{n=1}^{\infty}\lambda(n)B^2(n) \le \Big(\frac{V_1^2+V_2^2}{2}\Big)\sum_{n=1}^{\infty}\lambda(n)a^2(n). \tag{21}$$

We note that the inequalities obtained in (20) and (21) are the variants of Copson's inequality givin in (1) and (2).

Theorem 7. *Let $H, \lambda(n), \beta_j(n), a_j(n), \Lambda_j(n), A_j(n)$ be defined as in Theorem 5 for $j = 1,2$ and let $p_1, p_2 \ge 1$, $\sum_{n=1}^{\infty}\lambda(n)H^{p_1+p_2}(a_j(n))$ converge. If there exists $S_j \ge (p_1+p_2)/(p_1+p_2-1)$, such that*

$$(p_1+p_2)-1+\frac{(\beta_j(n+1)-\beta_j(n))\Lambda_j(n)}{\beta_j(n+1)\beta_j(n)\lambda(n)} \ge \frac{p_1+p_2}{S_j}, \quad \textit{for} \quad j=1,2,$$

then

$$\begin{aligned}\textstyle\sum_{n=1}^{\infty}\lambda(n)H^{p_1}\Big(\frac{A_1(n)}{\Lambda_1(n)}\Big)H^{p_2}\Big(\frac{A_2(n)}{\Lambda_2(n)}\Big) &\le \Big(\frac{p_1}{p_1+p_2}\Big)S_1^{(p_1+p_2)}\sum_{n=1}^{\infty}\lambda(n)H^{(p_1+p_2)}(a_1(n))\\ &\quad+\Big(\frac{p_2}{p_1+p_2}\Big)S_2^{(p_1+p_2)}\sum_{n=1}^{\infty}\lambda(n)H^{(p_1+p_2)}(a_2(n))\end{aligned} \tag{22}$$

Proof. By the elementary inequality (see[4])

$$p_1x^{p_1+p_2}+p_2y^{p_1+p_2}-(p_1+p_2)x^{p_1}y^{p_2} \ge 0,$$

where $x, y \ge 0$ and $p_1, p_2 > 0$ are real , we observe that

$$\begin{aligned}\sum_{n=1}^{\infty}\lambda(n)H^{p_1}\Big(\frac{A_1(n)}{\Lambda_1(n)}\Big)H^{p_2}\Big(\frac{A_2(n)}{\Lambda_2(n)}\Big) &\le \Big(\frac{p_1}{p_1+p_2}\Big)\sum_{n=1}^{\infty}\lambda(n)H^{(p_1+p_2)}\Big(\frac{A_1(n)}{\Lambda_1(n)}\Big)\\ &\quad+\Big(\frac{p_2}{p_1+p_2}\Big)\sum_{n=1}^{\infty}\lambda(n)H^{(p_1+p-2)}\Big(\frac{A_2(n)}{\Lambda_2(n)}\Big)\end{aligned}$$

Now a suitable application of Theorem 2 when $M = 1$ on the right side of the above inequality yields the required inequality in (22) and the proof of Theorem 7 is complete.

Theorem 8. *Let $\lambda(n), \beta_j(n), a_j(n), \Lambda_j(n), B_j(n)$ be defined as in Theorem 6 and let $p_1, p_2 \geq 1$, $\sum_{n=1}^{\infty} \lambda(n) a_j^{p_1+p_2}(n)$ converge. If there exists $T_j \geq p_1 + p_2$ such that*

$$1 - \frac{(\beta_j(n) - \beta_j(n-1))\Lambda_j(n-1)}{\beta_j(n)\beta_j(n-1)\lambda(n)} \geq \frac{p_1 + p_2}{T_j} \quad for\ j = 1, 2,$$

then

$$\sum_{n=1}^{\infty} \lambda(n) B_1^{p_1}(n) B_2^{p_2}(n) \leq \Big(\frac{p_1}{p_1 + p_2}\Big) T_1^{(p_1+p_2)} \sum_{n=1}^{\infty} \lambda(n) a_1^{p_1+p_2}(n) + \Big(\frac{p_2}{p_1 + p_2}\Big) T_2^{(p_1+p_2)} \sum_{n=1}^{\infty} \lambda(n) a_2^{p_1+p_2}(n)$$

Proof. By the same steps as in the proof of Theorem 7 with suitable modifications and replacing Theorem 2 by Theorem 4.

Remark 6. Theorem 2 in [12] is the special case of the Theorem 7 when $\beta_j(n) = 1$, $S_j = (p_1 + p_2)/(p_1 + p_2 - 1)$, for $j = 1, 2$, and if we take $H(u) = u$, $p_1 = p_2 = p$, $a_j(n) = a(n)$, $\Lambda_j(n) = \Lambda(n)$, $A_j(n) = A(n)$; $B_j(n) = B(n)$ in Theorem 7 and Theorem 8, then Theorem 7 and Theorem 8 reduce respectively to

$$\sum_{n=1}^{\infty} \lambda(n) \Big(\frac{A(n)}{\Lambda(n)}\Big)^{2p} \leq \Big(\frac{S_1^{2p} + S_2^{2p}}{2}\Big) \sum_{n=1}^{\infty} \lambda(n) a^{2p}(n) \tag{23}$$

and

$$\sum_{n=1}^{\infty} \lambda(n) B^{2p}(n) \leq \Big(\frac{T_1^{2p} + T_2^{2p}}{2}\Big) \sum_{n=1}^{\infty} \lambda(n) a^{2p}(n). \tag{24}$$

We note that the inequalities obtain in (23) and (24) are variants of copson's inequality given in (1) and (2).

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