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ORDER CONVERGENCE OF ORDER BOUNDED SEQUENCES IN RIESZ SPACES

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Abstract. We consider sequences $(x_n)_{n=1}^{\infty}$ in a Dedekind σ-complete Riesz space, satisfying a recursive relation

$$x_{n+p} \geq \sum_{j=1}^{p} \alpha_{nj} x_{n+p-j} \qquad \text{for } n = 1, 2, \cdots$$

where p is a given natural number and α_{nj} are nonnegative real numbers satisfying $\sum_{j=1}^{p} \alpha_{nj} = 1$. We obtain a sufficient condition on coefficients α_{nj} for which order boundedness of such a sequence $(x_n)_{n=1}^{\infty}$ implies its order convergence. In a particular case when $\alpha_{nj} = \alpha_j$ for all n and j, it is shown that every order bounded sequence satisfying the above recursive relation order converges if and only if natural numbers $j \leq p$ for which $\alpha_j > 0$, are relative prime.

1. Introduction

Throughout this paper we denote by N the set of natural numbers and by R the set of real numbers. For notation, terminology and basic results on Riesz spaces (vector lattices) we refer the reader to [1] and [2]. We recall here some definitions. A sequence $(x_n)_{n=1}^{\infty}$ in a Riesz space L is said to be *increasing* (respectively *decreasing*) if $x_n \leq x_{n+1}$ (respectively $x_{n+1} \leq x_n$) for all $n \in$N; it is *order bounded* if there exists an element $y \in L$ such that $|x_n| \leq y$ for all $n \in$ N, and *order convergent* if there exists an element $x \in L$ such that $|x_n - x| \leq v_n$ for some decreasing sequence $(v_n)_{n=1}^{\infty}$ satisfying $\inf_{n \in \mathrm{N}} V_n = 0$. A Riesz space L is said to be *Dedekind σ-complete* if for every increasing order bounded sequence $(x_n)_{n=1}^{\infty}$ in L there exists $\sup_{n \in \mathrm{N}} x_n$. It is easy to see that a Riesz space L is Dedekind σ-complete if and only if every order bounded increasing sequence in L order converges.

In this note we study order convergence of order bounded sequences $(x_n)_{n=1}^{\infty}$ satisfying a recursive relation

$$x_{n+p} \geq \sum_{j=1}^{p} \alpha_{nj} x_{n+p-j} \qquad \text{for } n = 1, 2 \cdots$$

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where $p \in \mathrm{N}$, $0 \leq \alpha_{nj} \in \mathrm{R}$, and $\sum_{j=1}^{p} \alpha_{nj} = 1$. We obtain a sufficient condition on coefficients α_{nj} for which order boundedness of such a sequence $(x_n)_{n=1}^{\infty}$ in a Dedekind σ-complete Riesz space implies its order convergence. In a particular case when $\alpha_{nj} = \alpha_j$ for all n and j, it is shown that every order bounded sequence satisfying the above recursive relation order converges if and only if natural numbers $j \leq p$ for which $\alpha_j > 0$, are relative prime.

Results. In order to shorten the formulation of our main result let us introduce the following set of functions. If $p \in \mathrm{N}$ and $\alpha_{nj} \in \mathrm{R}$, where $n = 1, 2 \ldots$ and $1 \leq j \leq p$, then for every triple $(\beta, m, n) \in \mathrm{R} \times \mathrm{N} \times \mathrm{N}$ denote by $S_\beta(m, n)$ the set of all functions $f : \{0, 1, \cdots, m\} \to \mathrm{N}$ such that $f(0) = n$ and

$$1 \leq f(k+1) - f(k) \leq p, \qquad \alpha_{f(k+1), f(k+1)-f(k)} \geq \beta$$

for all nonnegative integers $k < m$.

Theorem. *Let L be a Dedekind σ-complete Riesz space, and $(x_n)_{n=1}^{\infty}$ and order bounded sequence of its elements such that*

$$x_{n+p} \geq \sum_{j=1}^{p} \alpha_{nj} x_{n+p-j} \qquad \text{for all } n \in \mathrm{N}, \tag{1}$$

where p is a given natural number and α_{nj} are nonnegative real numbers satisfying $\sum_{j=1}^{p} \alpha_{nj} = 1$ for all $n \in \mathrm{N}$. If there exist a real number $\beta > 0$ and natural numbers M, N such that for every $n \geq N$ the set

$$R_\beta(M, n) = \{f(k) : f \in S_\beta(m, n), 0 \leq k \leq m \leq M\}$$

contains p consecutive natural numbers, then the sequence $(x_n)_{n=1}^{\infty}$ is order convergent.

Proof. Suppose that the numbers α_{nj} and the sequence $(x_n)_{n=1}^{\infty}$ satisfy all conditions of the theorem. Since $(x_n)_{n=1}^{\infty}$ is order bounded, the Dedekind σ-completeness of L implies that there exist elements

$$x = \sup_{n \in \mathrm{N}} \inf_{j \geq n} x_j, \qquad y = \inf_{n \in \mathrm{N}} \sup_{j \geq n} x_j.$$

Obviously $x \leq y$, and we have to show that $x = y$[2]. Suppose by contradiction that $x < y$ and assume without loss of generality that $x = 0$ (replacing x_n by $x_n - x$, if necessary). Let $\epsilon > 0$ and observe that there exists $m \in \mathrm{N}$ such that

$$u = (\inf_{j \geq m} x_j + \epsilon y)^+ > 0.$$

If there exists $k \in \mathrm{N}$ such that $x_j \wedge y \leq -u$ for all $j \geq k$, then $\sup_{j \geq k} x_j \wedge y \leq y - u$ which contradicts the definition of y. Hence, there exists a natural $n \geq \max\{m, N\}$ such that

$$v = (x_{n+p} \wedge y + u - y)^+ > 0.$$

If P is the projection onto the band of L generated by v, then $P(x_{n+p} \wedge y + u - y) = v$. Using also the inequality $u \leq \epsilon y$, it can be seen easily that

$$Px_{n+p} \geq v + (1-\epsilon)Py. \tag{2}$$

If $i \geq m$, then $(x_i + \epsilon y)^+ \geq u$ and therefore $(x_i + \epsilon y)^- \wedge u = 0$. Since $0 < v \leq u$, this implies $(x_i + \epsilon y)^- \wedge v = 0$. It follows that $P(x_i + \epsilon y)^- = 0$ and consequently $P(x_i + \epsilon y) \geq Pu \geq v$, hence

$$Px_i \geq v - \epsilon Py \qquad \text{for all } i \geq m. \tag{3}$$

We claim that

$$Px_{f(k)+p} \geq v + (\beta^k - \epsilon)Py \tag{4}$$

for all $f \in S_\beta(m,n)$ and $k \in \{0, \cdots, m-1\}$. Indeed, it follows from (2) that (4) holds for $k = 0$. We proceed the proof by induction on k. Assume that (4) holds for some nonnegative $k < m - 1$. Using (1) and (3) we get

$$\begin{aligned} Px_{f(k+1)+p} &\geq \sum_{j=1}^{p} \alpha_{f(k+1),j} Px_{f(k+1)+p-j} \\ &\geq \alpha_{f(k+1),f(k+1)-f(k)} Px_{f(k)+p} + (1 - \alpha_{f(k+1),f(k+1)-f(k)})(v - \epsilon Py) \\ &\geq v + (\alpha_{f(k+1),f(k+1)-f(k)}\beta^k - \epsilon)Py \geq v + (\beta^{k+1} - \epsilon)Py \end{aligned}$$

and the induction step is complete.

By assumption there exist p consecutive numbers

$$f_j(k_j) = l + j, \qquad j = 1, \cdots, p,$$

where $f_j \in S_\beta(m_j, n)$ and $0 \leq k_j \leq m_j \leq M$ for $j = 1, \cdots, p$. Note that $0 < \beta \leq 1$, and take $\epsilon = \beta^M$. The inequality (4) implies that

$$Px_{l+j+p} \geq v + (\beta^{k_j} - \epsilon)Py \geq v$$

holds for $j = 1, \cdots, p$. Using (1) we get easily $Px_i \geq v$ for all $i > l + p$. It follows that $\sup_{n \in \mathrm{N}} \inf_{j \geq n} Px_j \geq v$, which contradicts the equality $\sup_{n \in \mathrm{N}} \inf_{j \geq n} x_j = x = 0$.

The above theorem can be modified a little if a sequence $(x_n)_{n=1}^{\infty}$ satisfies

$$x_{n+p} = \sum_{j=1}^{p} \alpha_{nj} x_{n+p-j} \qquad \text{for all } n \in \mathrm{N}.$$

Namely, in this case $\inf_{1 \leq j \leq p} x_j \leq x_n \leq \sup_{1 \leq j \leq p} x_j$ holds for all $n \in \mathrm{N}$, hence the requirement of order boundedness is superfluous in the theorem.

The next result gives a sufficient condition for coeficients α_{nj} to satisfy the requirement of the theorem.

Proposition. *If there exists a subset $J \subseteq \{1, \cdots, p\}$ which consists of k relatively prime numbers $j_1, \cdots, j_k$ and satisfies*

$$\beta = \inf\{\alpha_{nj} : n \in \mathrm{N}, j \in J\} > 0, \tag{5}$$

then there exists $M \in \mathrm{N}$ such that for every $n \in \mathrm{N}$ the set $R_\beta(M,n)$ contains $p+1$ consecutive numbers.

Proof. Let $n \in N$. If $f : \{0, 1, \cdots, m\} \to \mathrm{N}$ satisfies $f(0) = n$ and $f(k+1) - f(k) \in J$ for all nonnegative integers $k < m$, then by (5) we have $f \in S_\beta(m,n)$. It follows that the set $\{f(k) : f \in S_\beta(m,n), 0 \leq k \leq m\}$ consists of all numbers of the form

$$n + \sum_{i=1}^{k} m_i j_i, \quad 0 \leq m_i \in \mathrm{Z}, \quad \sum_{i=1}^{k} m_i = m.$$

Since $j_1, \cdots, j_k$ are relatively prime, there exists integers $l_1, \cdots, l_k$ such that $\sum_{i=1}^{k} l_i j_i = 1$. Put

$$l = \max_{1 \leq i \leq k} |l_i|, \qquad M = 2kpl,$$

and note that for each nonnegative integer $q \leq p$ we have

$$m_i(q) = ql_i + pl \geq 0, \quad \sum_{i=1}^{k} m_i(q) \leq M.$$

It follows that for each $n \in \mathrm{N}$ the set $R_\beta(M,n)$ contains numbers

$$n + q + pl \sum_{i=1}^{k} j_i = n + \sum_{j=1}^{k} m_i(q) j_i,$$

where $q = 0, 1, \cdots, p$.

The proof of the above proposition can be shortened if we use the fact that the additive subsemigroup $S(J)$ of N generated by a subset $J = \{j_1, \cdots, j_k\}$ with relatively prime numbers $j_1, \cdots, j_k$, has finite complement in N. Indeed, since $S(J)$ consists of all sums of the form $\sum_{i=1}^{k} m_i j_i$, $0 \leq m_i \in \mathrm{Z}$, there exists $M \in \mathrm{N}$ such that for every $n \in \mathrm{N}$ the set $R_\beta(M,n)$ has finite complement in N.

In a special case of the above theorem and proposition we get the following result.

Corollary. *Let L be a nontrivial Dedekind σ-complete Riesz space, and $(x_n)_{n=1}^{\infty}$ a sequence of its elements such that*

$$x_{n+p} \geq \sum_{j=1}^{p} \alpha_j x_{n+p-j} \qquad \text{for all } n \in \mathrm{N}, \tag{6}$$

where the nonnegative real numbers α_j satisfy $\sum_{j=1}^{p} \alpha_j = 1$. Then the following statements are equivalent.

(i) *The natural numbers $j \leq p$ satisfying $\alpha_j > 0$ are relatively prime.*
(ii) *The sequence $(x_n)_{n=1}^{\infty}$ is order convergent if and only if it is order bounded.*

Proof. Take $\alpha_{nj} = \alpha_j$ for each $n \in \mathrm{N}$ and $j \in \{1, \cdots, p\}$, and put

$$J = \{j \in N : 1 \leq j \leq p, \alpha_j > 0\}.$$

If (i) holds, then J satisfies the condition of the proposition, hence (ii) follows from the theorem.

If (i) does not hold, then the greatest common divisor m of the numbers from J is greater then 1. For a nonzero $a \in L$, the sequence $(x_n)_{n=1}^{\infty}$ defined by

$$x_n = \begin{cases} a & \text{if } m|n \\ 0 & \text{otherwise} \end{cases}$$

satisfies (6). Since it is order bounded but not order convergent, (ii) does not hold.

An inspection of the proof shows that if the sequence $(x_n)_{n=1}^{\infty}$ of the above corollary satisfies

$$x_{n+p} = \sum_{j=1}^{p} \alpha_j x_{n+p-j} \qquad \text{for all } n \in \mathrm{N},$$

then it is order convergent if and only if the numbers of J are relatively prime.

The condition that L is Dedekind σ-complete cannot be dropped in the above theorem and corollary. Moreover, if L is not Dedekind σ-complete, there exists an order bounded increasing sequence in L which is not order convergent but obviously satisfies (1) and (6), hence the variant of the theorem and its corollary fails for such a space L.

References

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