TAMKANG JOURNAL OF MATHEMATICS
Volume 29, Number 1, Spring 1998

ANGULAR ESTIMATIONS OF CERTAIN ANALYTIC FUNCTIONS

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Abstract. The object of the present paper is to investigate some argument properties of certain analytic functions in the open unit disk. Our result contain some interesting corollaries as the special cases.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. If f and g are analytic in U, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if g is univalent in U, $f(0) = g(0)$ and $f(U) \subseteq g(U)$. Let $S(a, \alpha)$ denote the subclass of A consisting of functions which satisfy

$$|\frac{zf'(z)}{f(z)} - a| < a - \alpha \ (z \in U)$$

for some $\alpha (0 \leq \alpha < 1)$ and $a (a > \frac{1+\alpha}{2})$. The class $S(a, \alpha)$ was intrdouced by Sekine and Owa[8]. It is clear that, if a function f belongs to the class $S(a, \alpha)$, then

$$|\arg \frac{zf'(z)}{f(z)}| < \frac{\pi\beta}{2} \qquad (z \in U),$$

where

$$\beta = \frac{2}{\pi} Sin^{-1}(\frac{a-\alpha}{a}) \qquad (0 \leq \alpha < 1).$$

Furthermore, we note that for $\alpha = 0$, taking $a \to \infty$, the class $S(a, \alpha)$ is wellknown class of starlike functions with respect to the orign in U.

Received April 9, 1997.
1991 *Mathematics Subject Classification.* 30C45.
Key words and phrases. subordinate, angular estimations, starlike functions

The purpose of the present paper is to give some argument properties of analytic functions belonging to A. Our results have some intersting corollaries and include several previous results of Sakaguchi[7], Libera[1], Macgregor[2], Miller and Mocanu[3] and Nunokawa[6].

2. Main results

To esrablish our main results, we need the following lemmas.

Lemma 1([4]). *Let $h \in C$, the class of convex functions in U, and let $\lambda(z)$ be analytic in U with $Re\lambda(z) \geq 0$. If $p(z)$ is analytic in U and $p(0) = h(0)$, then*

$$p(z) + \lambda(z)zp'(z) \prec h(z) \ (z \in U)$$

implies

$$p(z) \prec h(z) \ (z \in U).$$

Lemma 2([5]). *Let $P(z)$ be analytic in U, $p(0) = 1$, $p(z) \neq 0$ in U and suppose that there exists a point $z_0 \in U$ such that*

$$|\arg p(z)| < \frac{\pi\eta}{2} \ for \ |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi\eta}{2},$$

where $\eta > 0$. Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\eta,$$

where

$$k \geq \frac{1}{2}(b + \frac{1}{b}) \ when \ arg \ p(z_0) = \frac{\pi\eta}{2}$$

and

$$k \leq -\frac{1}{2}(b + \frac{1}{b}) \ when \ arg \ p(z_0) = -\frac{\pi\eta}{2}$$

where

$$p(z_0)^{\frac{1}{\eta}} = \pm ib \ (b > 0).$$

With the help of Lemma 1 and Lemma 2, we now derive

Teorem 1: *Let $f \in A$ and $g \in S(a, \alpha)$. If*

$$|arg((1-\gamma)\frac{f(z)}{g(z)} + \gamma\frac{f'(z)}{g'(z)} - \beta)| < \frac{\pi\delta}{2}(\gamma \geq 0, 0 \leq \beta < 1, 0 < \delta \leq 1),$$

then

$$|\arg(\frac{f(z)}{g(z)} - \beta)| < \frac{\pi\eta}{2},$$

where $\eta(0 < \eta \leq 1)$ is the solution of the equation

$$\delta = \eta + \frac{2}{\pi}Tan^{-1}(\frac{\gamma\eta sin\frac{\pi}{2}(1 - t(a,\alpha))}{2a - \alpha + \gamma\eta cos\frac{\pi}{2}(1 - t(a,\alpha))}) \tag{2.1}$$

and

$$t(a,\alpha) = \frac{2}{\pi}Sin^{-1}(\frac{a-\alpha}{a}) \quad (0 \leq \alpha < 1). \tag{2.2}$$

Proof. Let us put

$$p(z) = \frac{1}{1-\beta}(\frac{f(z)}{g(z)} - \beta) \tag{2.3}$$

Then $p(z)$ is analytic in U with $p(0) = 1$. Differentiating (2.3) logarithmically, we have

$$\frac{1}{1-\beta}(\frac{f'(z)}{g'(z)} - \beta) = p(z) + \frac{g(z)}{zg'(z)}zp'(z).$$

Hence we obtain

$$(1-\gamma)\frac{f(z)}{g(z)} + \gamma\frac{f'(z)}{g'(z)} - \beta = (1-\beta)\big(p(z) + \frac{\gamma g(z)}{zg'(z)}zp'(z)\big).$$

Applying the assumption and Lemma 1 with $\lambda(z) = \frac{\gamma g(z)}{zg'(z)}$, we see that $p(z) \neq 0$ in U.

If there exists a point $z_0 \in U$ such that

$$|\arg p(z)| < \frac{\pi\eta}{2} \text{ for } |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi\eta}{2},$$

then, from Lemma 2, we have

$$\frac{z_0p'(z_0)}{p(z_0)} = ik\eta,$$

where

$$k \geq \frac{1}{2}(b + \frac{1}{b}) \text{ when } \arg p(z_0) = \frac{\pi\eta}{2}$$

and

$$k \leq -\frac{1}{2}(b + \frac{1}{b}) \text{ when } \arg p(z_0) = -\frac{\pi\eta}{2},$$

where

$$p(z_0)^{\frac{1}{\eta}} = \pm ib \ (b > 0).$$

Since $g \in S(a, \alpha)$ for $a > \frac{1+\alpha}{2} (0 \leq \alpha < 1)$,

$$\frac{zg'(z)}{g(z)} = re^{i\frac{\pi\phi}{2}},$$

where

$$\alpha < r < 2a - \alpha$$

and

$$-\frac{2}{\pi} Sin^{-1}(\frac{a-\alpha}{a}) < \phi < \frac{2}{\pi} Sin^{-1}(\frac{a-\alpha}{a}) \quad (0 \leq \alpha < 1).$$

Suppose that $p(z_0)^{\frac{1}{\eta}} = ib$ $(b > 0)$. Then we have

$$\begin{aligned}
&arg((1-\gamma)\frac{f(z_0)}{g(z_0)} + \gamma\frac{f'(z_0)}{g'(z_0)} - \beta) \\
&= \arg\left((1-\beta)p(z_0)(1 + \frac{\gamma g(z_0)}{z_0 g'(z_0)} \frac{z_0 p'(z_0)}{p(z_0)})\right) \\
&= \arg p(z_0) + \arg\left(1 + \frac{\gamma g(z_0)}{z_0 g'(z_0)} \frac{z_0 p'(z_0)}{p(z_0)}\right) \\
&= \frac{\pi\eta}{2} + \arg\left(1 + \gamma(re^{i\frac{\pi\phi}{2}})^{-1} i\eta k\right) \\
&= \frac{\pi\eta}{2} + Tan^{-1}(\frac{\gamma\eta k sin\frac{\pi}{2}(1-\phi)}{r + \gamma\eta k cos\frac{\pi}{2}(1-\phi)}) \\
&\geq \frac{\pi\eta}{2} + Tan^{-1}(\frac{\gamma\eta sin\frac{\pi}{2}(1 - t(a,\alpha))}{2a - \alpha + \gamma\eta cos\frac{\pi}{2}(1 - t(a,\alpha))}) \\
&= \frac{\pi\delta}{2},
\end{aligned}$$

where $t(a, \alpha)$ and δ are given by (2.2) and (2.1), respectively. This is a contradiction to the assumption of our theorem.

Next, suppose $p(z_0)^{\frac{1}{\eta}} = -ib$ $(b > 0)$. Applying the same methad as the above, we have

$$\begin{aligned}
&\arg\left((1-\gamma)\frac{f(z_0)}{g(z_0)} + \gamma\frac{f'(z_0)}{g'(z_0)} - \beta\right) \\
&\leq -\frac{\pi\eta}{2} - Tan^{-1}(\frac{\gamma\eta sin\frac{\pi}{2}(1 - t(a,\alpha))}{2a - \alpha + \gamma\eta cos\frac{\pi}{2}(1 - t(a,\alpha))}) \\
&= -\frac{\pi\delta}{2},
\end{aligned}$$

where $t(a, \alpha)$ and δ are given by (2.2) and (2.1), respectively, which contradicts the assumption. Therefore we complete the proof of our theorem.

Taking $\gamma = 1$ in Theorem 1, we have

Corrllary 1. *Let $f \in A$ and $g \in S(a, \alpha)$, if*

$$|\arg\left(\frac{f'(z)}{g'(z)} - \beta\right)| < \frac{\pi\delta}{2} \ (0 \leq \beta < 1, 0 < \delta \leq 1),$$

then

$$|\arg\left(\frac{f(z)}{g(z)} - \beta\right)| < \frac{\pi\eta}{2},$$

where $\eta (0 < \eta \leq 1)$ is the solution of the equation

$$\delta = \eta + \frac{2}{\pi} Tan^{-1}\left(\frac{\eta sin\frac{\pi}{2}(1 - t(a, \alpha))}{2a - \alpha + \eta cos\frac{\pi}{2}(1 - t(a, \alpha))}\right) \tag{2.4}$$

and $t(a, \alpha)$ is given by (2.2).

Remark 1. Taking $\alpha = 0$, $a \to \infty$, $\gamma = 1$ and $\delta = 1$ is Corollary 1, we obtain the corresponding results of Sakaguchi[5], Libera[1], MacGregor[2] and Miller and Mocanu[3].

Putting $a = 1$, $\alpha \to 1$, $\beta = 0$ and $g(z) = z$ in Theorem 1, we get

Corollary 2. *Let $f \in A$. If*

$$|\arg\left((1 - \gamma)\frac{f(z)}{z} + \gamma f'(z)\right)| < \frac{\pi\delta}{2} \ (\gamma \geq 0, 0 < \delta \leq 1),$$

then

$$|\arg \frac{f(z)}{z}| < \frac{\pi\eta}{2},$$

where $\eta (0 < \eta \leq 1)$ is the solution of the equation

$$\delta = \eta + \frac{2}{\pi} Tan^{-1} \gamma\eta.$$

Similarly, we have

Theorem 2. *Let $f \in A$ and $g \in S(a, \alpha)$. If*

$$|\arg\left(\beta - \left((1 - \gamma)\frac{f(z)}{g(z)} + \gamma\frac{f'(z)}{g'(z)}\right)\right)| < \frac{\pi\delta}{2} (\gamma \geq 0, \beta > 1, 0 < \delta \leq 1),$$

then

$$|\arg\left(\beta - \frac{f(z)}{g(z)}\right)| < \frac{\pi\eta}{2},$$

where $\eta (0 < \eta \leq 1)$ is the solution of the equaiton (2.1).

Remark 2. Putting $\alpha = 0$, $a \to \infty$ and $\gamma = 1$ in Theorem 1 and Teorem 2, we have the corresponding results of Nunokawa[6].

Next, we prove

Theorem 3. *Let $f \in A$. If*

$$|\arg \frac{zf'(z)}{f^{1-m}(z)z^m}| < \frac{\pi\delta}{2} \ (0 < \delta \leq 1, m \in N),$$

then

$$|\arg (\frac{f(z)}{z})^m| < \frac{\pi\eta}{2},$$

where $\eta (0 < \eta \leq 1)$ is the solution of the equation

$$\delta = \eta + \frac{2}{\pi} Tan^{-1} \frac{\eta}{m}$$

Proof. Let us put

$$p(z) = (\frac{f(z)}{z})^m.$$

Then $p(z)$ is analytic in U with $p(0) = 1$ and we have

$$\frac{zf'(z)}{f^{1-m}(z)z^m} = p(z) + \frac{1}{m} zp'(z).$$

By the assumption and Lemma 1, we can see that $p(z) \neq 0$ in U. The remaining part of the proof is similar to that of Theorem 1 and so we omit it.

Finally, we have

Theorem 4. *Let $f \in A$. If*

$$|\arg \frac{zf'(z)}{f^{1-u}(z)z^u}| < \frac{\pi\delta}{2} \ (u > 0, 0 < \delta \leq 1),$$
$$|\arg \frac{zF'(z)}{F^{1-u}(z)z^u}| < \frac{\pi\eta}{2},$$

where F is given by

$$F^u(z) = \frac{u+c}{z^c} \int_0^z t^{c-1} f^u(t) dt \quad (c > 0) \tag{2.5}$$

with $F(z) \neq 0$ in $U - \{0\}$ and $\eta (0 < \eta \leq 1)$ is the solution of the equation

$$\delta = \eta + \frac{2}{\pi} Tan^{-1} \frac{\eta}{u+c} \tag{2.6}$$

Proof. From the definition of (2.5), we have

$$u \frac{zF'(z)}{F^{1-u}(z)} + cF^u(z) = (u+c) f^u(z).$$

Putting

$$p(z) \doteq \frac{T(z)}{S(z)},$$

where $T(z) = F'(z)z^{u+c}/(F(z)/z)^{1-u}$ and $S(z) = z^{u+c}$, we have

$$\frac{T'(z)}{S'(z)} = p(z) + \frac{1}{u+c}zp'(z) = \frac{zf'(z)}{f^{1-u}(z)z^u}$$

Applying Lemma 1 and the proof of Theorem 1, we obtain

$$|\arg \frac{zF'(z)}{F^{1-u}(z)z^u}| < \frac{\pi\eta}{2},$$

where $\eta(0 < \eta \leq 1)$ is the solution of the equation (2.6).

Acknowledgements

This work was partially supported by Non Directed Research Fund, Korea Research Foundation, 1996 and the Basic Science Research Program, Ministry of Education, Project No. BSRI-97-1440.

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