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DECOMPOSITIONS OF $K_{m,n}$ INTO 4-CYCLES AND 8-CYCLES

CHIN-MEI FU AND WEN-CHUNG HUANG

Abstract. In this paper it is shown that G can be decomposed into p copies of C_4 and q copies of C_8 for each pair of nonnegative integers p and q which satisfies the equation $4p+8q=|E(G)|$, where —E(G) is the number of edges of G, when

(1) $G=K_{m,n}$, the complete bigartite graph, if m and n are even,
(2) $G=K_{m,n}-F$, $K_{m,n}$ with 1-factor removed, if $m=n\equiv 1 \pmod 4$, and
(3) $G=K_{m,n}-(F\cup C_6)$, $K_{m,n}$ with 1-factor and one 6-cycle removed, if $m=n\equiv 3$ (mod 4).

Let m, n, and r be positive integers. Let $K_{m,n}$ denote the complete bipartite graph. A 2r-cycle, C_{2r}, is an elementary cycle of length 2r and will denoted by the sequence of its vertices $(x_1, x_2, \ldots x_{2r})$. A graph G can be decomposed into C_4 and C_8 if it is possible to partiton the edge-set of G into of cycles of length 4 or 8. If the edge-set of G can be decomposed into p copies of C_4 and q copies of C_8, then G will be denoted by $G=pC_4+qC_8$.

The problem of the existence of a decomposition of the complete graph K_n(the complete graph from which a 1-factor has been removed, K_n-F) into cycles of different lengths has been investigated several times (see[1] for reference). In [5], we can see the decomposition of the complete bipartite graph $K_{m,n}$ into cycles of length $2k$. In this paper, we show that G can be decomposed into p copies of C_4 and q copies of C_8 for each pair of nonegative integers p and q which satisfies the equation $4p+8q=|E(G)|$, where $|E(G)|$ is the number of edges of G, when $G=K_{m,n}$ if m and n are even, $G=K_{m,n}-F$ if $m=n\equiv 1 \pmod 4$, and $G=K_{m,n}-(F\cup C_6)$ if $m=n\equiv 1 \pmod 4$.

For convenience, we define $D(G)=\{(p,q)\mid$ p,q are nonnegative integers, $G=pC_4+qC_8\}$, and $S_i=\{(p,q)\mid$ p,q are nonnegative integers, $p+2q=i\}$, where i is a positive integer. Then $D(K_{m,n})\subseteq S_{mn/4}$.

Since the necessary and sufficient condition for the equation $4p+8q=mn$ having nonnegative integer solution is that mn is the multiple of 4.

Main Theorem. *Let s and t be positive integers. Then* (1) $D(K_{2s,2t})=S_{st}$, (2) $D(K_{4s+1,4s+1}-F)=S_{s(4s+1)}$, *where F is a 1-factor in $K_{4s+1,4s+1}$, and* (3) $D(K_{4s+3,4s+3}-(F\cup C_6))=S_{s(4s+5)}$, *where $F\cup C_6$ is a 1-factor and one cycle of length 6 in $K_{4s+3,4s+3}$.*

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Let A and B be two sets of ordered pairs, we define that $A+B = \{(a_1+b_1, a_2+b_2) \mid (a_1, a_2) \in A, (b_1, b_2) \in B\}$, and $2 \cdot A = A + A$. Let $V(K_{m,n}) = \{x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n\}$.

Lemma 1. *Let a and b be any positive integers. If one of a,b is even, then $S_{a+b} = S_a + S_b$; if a and b are odd, then $S_{a+b} = (S_a + S_b) \cup \{(0, (a+b)/2)\}$.*

Proof. If a and b are odd, then the minimum value of p in S_a and S_b is 1, but the minimum value of p is 0 in S_{a+b}.

Next, we will show that $D(K_{2s,2t}) = S_{st}$.

Lemma 2. $D(K_{2,2t}) = \{(t, 0) \mid t$ *is a positive integer*$\}$.

Proof. For each positive integer t, $K_{2,2t}$ can be decomposed into t copies of $K_{2,2}$ and $K_{2,2} = C_4$.

It is easy to get the decomposition of $K_{4,4}$.

Lemma 3. $D(K_{4,4}) = \{(4, 0), (0, 2)\}$.

Lemma 4. *Let t be positive integer, $t \geq 3$, then $D(K_{4,2t}) = S_{2t}$.*

Proof.

(1) $t = 3$. $K_{4,6}$ can be viewed as adding two vertices and two 4-cycles to $K_{4,4}$, thus $K_{4,6} = 6C_4 = 2C_4 + 2C_8$. $K_{4,6} = 4C_4 + C_8$ for which we use the cycles $(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4)$, (x_1, y_2, x_4, y_5), (x_1, y_3, x_2, y_6), (x_2, y_4, x_3, y_5), and (x_3, y_1, x_4, y_6), and $K_{4,6} = 3C_8$ for which we take the 8-cycles $(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4)$,$(x_1, y_3, x_2, y_4, x_3, y_5, x_4, y_6)$, and $(x_1, y_5, x_2, y_6, x_3, y_1, x_4, y_2)$. Hence $D(K_{4,6}) = S_6$.

(2) $t = 4$. $K_{4,8}$ can be viewed as an edge-disjoint union of $K_{4,2}$ and$K_{4,6}$. Then we get $\{(8,0), (6,1), (4,2), (2,3)\} \subset D(K_{4,8})$. Since $K_{4,8}$ can also be decomposed into two copies of $K_{4,4}$, that means $(0, 4) \in D(K_{4,8})$. Thus $D(K_{4,8}) = S_8$.

(3) $t = 5$. Since $D(K_{4,10}) \supseteq D(K_{4,4}) + D(K_{4,6}) = S_{10}$, it means that $D(K_{4,10}) = S_{10}$.

(4) $t \geq 6$, $K_{4,2t}$ can be viewed as an edge-disjoint union of $K_{4,2t-6}$ and $K_{4,6}$. Then $D(K_{4,2t}) \supseteq D(K_{4,2t-6}) + D(K_{4,6})$. If $D(K_{4,2t-6}) = S_{2(t-3)}$, then by Lemma 1, $D(K_{4,2t}) \supseteq S_{2(t-3)} + S_6 = S_{2t}$. Thus the conclusion can be done recursively.

Theorem 5. *Let s, t be positive integers, $t \geq 3$. Then $D(K_{4s,2t}) = S_{2st}$.*

Proof. We can think of $K_{4s,2t}$ as an edge-disjoint union of s copies of $K_{4,2t}$.

Lemma 6. *Let t be positive integer, $t \geq 3$, then $D(K_{6,2t}) = S_{3t}$.*

Proof. Now $D(K_{6,6}) \supseteq D(K_{4,6}) + D(K_{2,6}) = S_9 \backslash \{(1, 4)\}$. And $C_4 + 4C_8$ is given by $(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4)$, $(x_1, y_5, x_2, y_6, x_5, y_2, x_6, y_3)$, $(x_1, y_2, x_4, y_1, x_3, y_5, x_6, y_6)$, $(x_3, y_6, x_4, y_5, x_5, y_1, x_6, y_4)$,$(x_2, y_3, x_5, y_4)$. For $t \geq 4, D(K_{6,2t}) \supseteq D(K_{6,2t-4}) + D(K_{6,4})$. By

Lemma 4, we have $D(K_{4,6}) = S_6$, and by Lemma 1, $S_{3(t-2)} + S_6 = S_{3t}$, then we can recursively obtain the conclusion.

Theorem 7. *Let s, t be positive integers, $t \geq 2$. $D(K_{4s+2,2t}) = S_{t(2s+1)}$.*

Proof. For $s \geq 2, t \geq 3, D(K_{4s+2,2t}) \supseteq D(K_{4s-4,2t}) + D(K_{6,2t})$. And we have $D(K_{4s-4,2t}) = S_{t(2s-2)}$ and $D(K_{6,2t}) = S_{3t}$. By Lemma 1, $S_{t(2s-2)} + S_{3t} = S_{t(2s+1)}$. Combine Lemma 6, $D(K_{4s+2,2t}) = S_{t(2s+1)}$, for $s \geq 1, t \geq 2$.

From Theorem 5 and 7, we have shown that

Theorem 8. *Let s, t be positive integers, $s \geq 2$ and $t \geq 3$. Then $D(K_{2s,2t}) = S_{st}$. That is $K_{2s,2t}$ can be decomposed into p copies of C_4 and q copies of C_8 for each pair of nonnegative integers p and q which satisfies the equation $4p + 8q = 4st$.*

Next, we consider the decomposition of $K_{4s+1,4s+1} - F$. We will show that $D(K_{4s+1,4s+1} - F) = S_{s(4s+1)}$.

Theorem 9. *Let s be positive integer, then $D(K_{4s+1,4s+1} - F) = S_{s(4s+1)}$, where F is a 1-factor of $K_{4s+1,4s+1}$.*

Proof.

(1) $s = 1$. The decomposition of $K_{5,5} - F$ is given as follows. $5C_4$: (x_1, y_2, x_4, y_3), (x_3, y_1, x_5, y_2), (x_2, y_1, x_4, y_5), (x_2, y_3, x_5, y_4), and (x_1, y_4, x_3, y_5). $3C_4 + C_8$: $(x_1, y_2, x_5, y_4, x_3, y_1, x_2, y_3)$, (x_1, y_4, x_2, y_5), (x_3, y_2, x_4, y_5), and (x_4, y_1, x_5, y_3). $C_4 + 2C_8$: $(x_1, y_2, x_4, y_5, x_3, y_4, x_2, y_3)$, $(x_1, y_4, x_5, y_3, x_4, y_1, x_2, y_5)$, and (x_3, y_1, x_5, y_2). Thus $D(K_{5,5} - F) = S_5$.

(2) $s = 2$. $K_{9,9} - F$ can be viewed as an edge-disjoint union of two copies of $K_{5,5} - F$ and two copies of $K_{4,4}$. Then $D(K_{9,9} - F) \supseteq 2 \cdot D(K_{5,5} - F) + 2 \cdot D(K_{4,4}) = 2 \cdot S_5 + 2 \cdot \{(4,0), (0,2)\} = S_{18} \backslash \{(0,8)\}$. And $K_{9,9} - F$ can be decomposed into 9 copies of C_8 : $(x_1, y_2, x_4, y_5, x_3, y_4, x_2, y_3)$, $(x_1, y_4, x_9, y_1, x_2, y_9, x_4, y_6)$, $(x_1, y_5, x_9, y_6, x_7, y_8, x_3, y_7)$, $(x_1, y_8, x_4, y_7, x_2, y_6, x_3, y_9)$, $(x_2, y_5, x_7, y_1, x_6, y_7, x_5, y_8)$, $(x_3, y_1, x_4, y_3, x_8, y_4, x_5, y_2)$, $(x_5, y_1, x_8, y_2, x_6, y_4, x_7, y_3)$, $(x_5, y_6, x_8, y_7, x_9, y_8, x_6, y_9)$ and $(x_6, y_3, x_9, y_2, x_7, y_9, x_8, y_5)$. Thus $D(K_{9,9} - F) = S_{18}$.

(3) For $s \geq 3$, $D(K_{4s+1,4s+1} - F) \supseteq D(K_{4s-7,4s-7} - F) + D(K_{9,9} - F) + 2 \cdot D(K_{8,4s-8})$. By Theorem 5, $D(K_{8,4s-8}) = S_{4(2s-4)}$, and $D(K_{9,9} - F) = S_{18}$. By Lemma 1, we can obtain that $S_{(s-2)(4s-7)} + S_{18} + 2 \cdot S_{4(2s-4)} = S_{s(4s+1)}$. then the theorem is proved recursively.

The degree of each vertex in $K_{4s+3,4s+3}$ is odd and the number of edges is not a multiple of 4, Thus we need to remove at least one factor such that the degree of each vertex is even, and take away one 6-cycle such that the number of edges in $K_{4s+3,4s+3} - (F \cup C_6)$ is a multiple of 4.

Theorem 10. *Let s be positive integer, then $D(K_{4s+3,4s+3} - (F \cup C_6)) = S_{s(4s+5)}$, where F is a 1-factor of $K_{4s+3,4s+3}$.*

Proof.

(1) $s = 1$. We can think $K_{7,7} - (F \cup C_6)$ as an edge-disjoint union of $K_{5,5} - F$ and two copies of $K_{2,4}$. Then $D(K_{7,7} - (F \cup C_6)) \supseteq D(K_{5,5} - F) + 2 \cdot D(K_{2,4}) = S_5 + 2 \cdot \{(2,0)\} = S_9 \backslash \{(1,4),(3,3)\}$. $3C_4 + 3C_8$ can be obtained: $(x_1, y_2, x_5, y_1, x_3, y_6, x_2, y_5)$, $(x_1, y_3, x_4, y_2, x_6, y_1, x_2, y_7)$, $(x_1, y_4, x_3, y_2, x_7, y_1, x_4, y_6)$, (x_5, y_3, x_6, y_4), (x_2, y_3, x_7, y_4), and (x_3, y_5, x_4, y_7). And $C_4 + 4C_8$: $(x_1, y_2, x_5, y_1, x_3, y_6, x_2, y_5)$, $(x_1, y_6, x_4, y_3, x_5, y_4, x_2, y_7)$, $(x_1, y_3, x_6, y_2, x_4, y_5, x_3, y_4)$, $(x_2, y_1, x_4, y_7, x_3, y_2, x_7, y_3)$, and (x_6, y_1, x_7, y_4).

(2) $s = 2$. We can view $K_{11,11} - (F \cup C_6)$ as an edge-disjoint union of $K_{7,7} - (F \cup C_6)$, $K_{5,5} - F$ and two copies of $K_{4,6}$. Then $D(K_{11,11} - (F \cup C_6)) \supseteq D(K_{7,7} - (F \cup C_6)) + D(K_{5,5} - F) + 2 \cdot D(K_{4,6}) = S_9 + S_5 + 2 \cdot S_6 = S_{26} \backslash \{(0,13)\}$. $13C_8$ is defined as follows: $(x_1, y_2, x_5, y_4, x_3, y_1, x_2, y_3)$, $(x_1, y_4, x_9, y_1, x_8, y_7, x_6, y_5)$, $(x_1, y_6, x_{10}, y_3, x_4, y_8, x_2, y_7)$, $(x_2, y_4, x_7, y_3, x_6, y_2, x_4, y_5)$, $(x_4, y_1, x_5, y_3, x_8, y_5, x_3, y_6)$, $(x_2, y_6, x_7, y_2, x_9, y_7, x_5, y_9)$, $(x_1, y_8, x_5, y_6, x_{11}, y_2, x_3, y_9)$, $(x_1, y_{10}, x_7, y_8, x_{11}, y_7, x_3, y_{11})$, $(x_2, y_{10}, x_6, y_4, x_{10}, y_7, x_4, y_{11})$, $(x_3, y_8, x_{10}, y_1, x_7, y_9, x_4, y_{10})$, $(x_5, y_{10}, x_8, y_6, x_9, y_8, x_6, y_{11})$, $(x_6, y_9, x_8, y_{11}, x_7, y_5, x_{11}, y_1)$, and $(x_8, y_2, x_{10}, y_5, x_9, y_3, x_{11}, y_4)$.

(3) For $s \geq 3$. Let $G = K_{4s+3,4s+3} - (F \cup C_6)$. G can be viewed as an edge-disjoint union of $K_{11,11} - (F \cup C_6)$, $K_{4s-7,4s-7} - F$, and two copies of $K_{10,4s-8}$. Then $D(G) \supseteq D(K_{4s-7,4s-7} - F) + D(K_{11,11} - (F \cup C_6)) + 2 \cdot D(K_{10,4s-8})$. By Theorem 7, $D(K_{10,4s-8}) = S_{5(2s-4)}$, and by Theorem 9, $D(K_{4s-7,4s-7} - F) = S_{(s-2)(4s+1)}$. And by Lemma 1, $S_{(s-2)(4s+1)} + S_{26} + S_{5(2s-4)} = S_{s(4s+5)}$. Therefore $D(G) = S_{s(4s+5)}$.

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Department of Mathematics, Tamkang University, Tamsui, Taipei Shien, Taiwan, Republic of China.

E-mail address:cmfu@mail.tku.edu.tw