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DVORETZKY-ROGERS THEOREM FOR SEQUENCE SPACES WITH $\sigma\mu$-TOPOLOGY

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Abstract. In this article Dvoretzky-Rogers theorem has been established for the sequence spaces equipped with $\sigma\mu$-topology.

The famous classical theorem of Dvoretzky-Rogers asserts that if E is a normed space for which $\ell^1(E) = \ell^1\{E\}$ (or equivalently, $\ell^1 \otimes_\varepsilon E \simeq \ell^1 \otimes_\pi E$), then E is of finite dimension (cf. [10], p.67). This property also remains preserved for any $\ell^p(1 < p < \infty)$ in place of ℓ^1 (cf [6], p.104 and [2] Corollary 5.5). In this context, De Grande-De Kimpe [3] provides an extension of Dvoretzky-Rogers theorem for perfect Banach sequence spaces and Andreu [1] brings forth the validity of the aforementioned theorem for any echelon space of order $p(1 < p < \infty)$ or order (p,q). It has been investigated that the result remains still true when one replaces ℓ^1 by any non-nuclear perfect sequence space having the normal topology (cf. [12]).

As a generalization of normal topology Ruckle [13] considers the $\sigma\mu$-*topology* associated with the sequence space μ on an arbitrary sequence space λ. This $\sigma\mu$-topology on λ is defined by the family $\{p_{y,z} : y \in \lambda^\mu, z \in \mu^\times\}$ of semi-norms, where

$$\lambda^\mu = \{y \in \omega : yx \in \mu, \quad \forall x \in \lambda\}$$

and

$$p_{y,z}(x) = \sum_{n=1}^{\infty} |x_n y_n z_n|, \quad x \in \lambda$$

(ω denotes the space of all scalar sequences)

Note. For $\mu = \ell^1$, we obtain $\lambda^\mu = \lambda^\times$, $\mu^x = \ell^\infty$ and $\sigma\mu$-topology on λ becomes the normal topology $\eta(\lambda, \lambda^\times)$. Furthermore, it is easily observed that this μ-dual λ^μ enoelops in particular, the well known α-, β-and γ-duals (cf. [14]).

The sequence space λ is said to be μ-perfect if $\lambda = \lambda^{\mu\mu} = (\lambda^\mu)^\mu$; where

$$\lambda^{\mu\mu} = \{z \in \omega : zy \in \mu, \quad \forall y \in \lambda^\mu\}$$

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Theorem F. *Suppose μ is a Hilbert space having a monotone normalized Schauder basis. Then $\Lambda_\mu(p)$ is nuclear if and only if $\Lambda(P)$ is nuclear.*

Proof. $\Lambda_\mu(P)$ is nuclear if and only if to each $a \in P$, there corresponds a $b \in P$, $(b \geq a)$ such that the canonical map $\hat{K}_a^b : \hat{\Lambda}_\mu(P;b) \longrightarrow \hat{\Lambda}_\mu(P;a)$ is nuclear ($\wedge$-denotes completion). One can identify the quotient space $\Lambda_\mu(p;a) = \Lambda(P)/\ker p_a$ with

$$\mu_a = \{x \in \mu : x_n = 0 \quad \text{for } n \text{ where } a_n = 0\}$$

via the unique extension to the isometrical isomorphism $\hat{\psi}_a$ of the embedding

$$\psi_a : \Lambda_\mu(P;a) \longrightarrow \mu_a$$

where

$$\psi_a(x) = \{a_n x_n\}, \quad x \in \Lambda_\mu(P).$$

Then

$$D_a^b = \hat{\psi}_a \; o \; K_a^b \; o \; \hat{\psi}_b^{-1}$$

is a diagonal map on μ, determined by the sequence $\{a_n/b_n\}$. In view of the observation made in page 144 in [16], K_a^b is nuclear if and only if D_a^b is nuclear and by the Theorem 8.3.3 in [10] this is equivalent to the fact that

$$\{\alpha_n(D_a^b)\} \in \ell^1$$

where α_n denotes the n-th approximation numbers. Hence by lemma 3.3 in [7], D_a^b is nuclear if and only if

$$\sum_{n\geq 1} a_n/b_n < \infty$$

i.e., $P \subseteq P\ell^1$. By the Grothendieck-Pietsch Criterion, this condition is equivalent to the nuclearity of $\Lambda(P)$ (cf., [10], Theorem 6.1.2).

Remarks. In view of Theorem F, proceeding in a similar way as in the case of Theorem D, one can obtain first the analogous of Corollary B and then prove that; for a normed space E, the following are equivalent:

(*i*) $\Lambda_\mu(P)(E) \simeq \Lambda_\mu(P)\{E\}$,
(*ii*) $\Lambda_\mu(P)[E] \simeq \Lambda_\mu(P)\{E\}$,
(*iii*) $\Lambda_\mu(P) \otimes_\varepsilon E \simeq \Lambda_\mu(P) \otimes_\pi E$,
(*iv*) $\Lambda_\mu(P)\tilde{\otimes}_\varepsilon E \simeq \Lambda_\mu(P)\tilde{\otimes}_\pi E$,
(*v*) $\Lambda_\mu(P)$ is nuclear or E is finite dimensional.

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and $C_j^n = 0$ if $j > n$. Since a review of the structure of C^n reveals that C^n belongs to $\lambda(E)$ for all n, it follows from (*) that

$$r\left\{\sum_{j=1}^{n} \|C_j\| \frac{|a_j y_j|}{|b_j z_j|}\right\} \leq \sup_{u\in U^\circ} \left\{\sum_{j=1}^{n} |< C_j, u >|\right\}$$
$$\leq \sup_{u\in U^\circ} \left\{\sum_{j=1}^{\infty} |< C_j, u >|\right\}$$

and consequently

$$(+) \qquad r\left\{\sum_{j=1}^{\infty} \|C_j\| \frac{|a_j y_j|}{|b_j z_j|}\right\} < \infty, \qquad \forall C \in \ell^1(E).$$

Applying the Dvoretzky-Rogers Lemma to (+), in view of Lemma 2.3.14[16] we conclude that

$$\left\{\frac{a_j y_j}{b_j z_j}\right\} \in \ell^2$$

and hence by Corollary B the space $(\lambda, \sigma\mu)$ is nuclear.

Given a Kothe set P and sequence space μ, we have the generalized Kothe space $\Lambda_\mu(P)$;

$$\Lambda_\mu(P) = \{x \in \omega : xy \in \mu, \quad \forall y \in P\}.$$

The natural locally convex topology on $\Lambda_\mu(P)$ is generated by the family

$$\{p_{a,y} : a \in P, y \in \mu^\times\}$$

of semi-norms, where

$$p_{a,y}(x) = \sum_{n\geq 1} |a_n x_n y_n|, \qquad x \in \Lambda_\mu(P).$$

For $\mu = \ell^1$, $\Lambda_\mu(P) = \Lambda(P)$ the Köthe space (cf. [10], p.97 and [16], p.190).

The following result in [15] characterizes the nuclearity of the space $\Lambda_\mu(P)$.

Theorem E. *$\Lambda_\mu(P)$ is nuclear if and only if to each $a \in P$ and $y \in \mu^\times$, there correspond $b \in P$ and $z \in \mu^\times$ such that*

$$\left\{\frac{a_n y_n}{b_n z_n}\right\} \in \ell^1$$

To strengthen the above result we assert that, nuclearity of a Köthe space $\Lambda(P)$ is synonymous with the nuclearity of the $\Lambda_\mu(P)$ if μ is a Hilbert space with a monotone normalized Schauder basis. Precisely, we have the following

Lemma C. *Let $(E, \|\ \|)$ be an infinite dimensional normed space and let $\delta = \{\delta_n\}$ be an element of ℓ^2. Then there is an $x = (x_n) \in \ell^1(E)$ with $\|x_n\| = |\delta_n|$ for all $n \in \mathbb{N}$.*

Now, for the sequence spaces equipped with $\sigma\mu$-topology we present Dvoretzky-Rogers theorem, which is basically contained in

Theorem D. *For a normed space E, the following are equivalent:*
(i) $\lambda(E) \simeq \lambda\{E\}$,
(ii) $\lambda[E] \simeq \lambda\{E\}$,
(iii) $\lambda \otimes_\varepsilon E \simeq \lambda \otimes_\pi E$,
(iv) $\lambda \tilde{\otimes}_\varepsilon E \simeq \lambda \tilde{\otimes}_\pi E$,
(v) *$(\lambda, \sigma\mu)$ is nuclear or E is finite dimensional.*

Proof. (i) $\Rightarrow$ (ii): since $\lambda(E)$ is provided with the ε-topology and $\lambda\{E\} \subset \lambda(E) \subset \lambda[E]$, we need only to establish that $\lambda[E] \subset \lambda\{E\}$. Let $x = (x_j)$ be an element in $\lambda[E]$. Then every $x^{(n)}$ of x is in $\lambda(E)$. By the hypothesis, given a $\in \lambda^\mu$ and $y \in \mu^\times$ there exist $b \in \lambda^\mu$, $z \in \mu^\times$ and a real number $r > 0$ such that

$$\begin{aligned} r\pi_{a,y}(x^{(n)}) &= r\left\{\sum_{j=1}^{n} \|x_j\|\ |a_j y_j|\right\} \\ &\leq \sup_{u \in \mathrm{U}^\circ} \left\{\sum_{j=1}^{n} |< x_j, u > b_j z_j|\right\} \\ &= \varepsilon_{b,z}(x^{(n)}). \end{aligned}$$

Since this holds for all n, we obtain

$$r\pi_{a,y}(x) \leq \varepsilon_{b,z}(x) < \infty.$$

(ii) $\Rightarrow$ (i) is obvious. Also (iii) $\Leftrightarrow$ (ii) is clear and (v) $\Rightarrow$ (iii) follows from Theorem 4.1 [16] (which is a well known Grothendieck's result (Theorem 7.3.8 [10])). (iii) $\Rightarrow$ (i) follows by an argument analogous to that of [9], p.197 and p. 291.

(i) $\to$ (v). Suppose E is infinite dimensional. Given a $\in \lambda^\mu$ and $y \in \mu^\times$ there exist $b \in \lambda^\mu$, $z \in \mu^\times$ and $r > 0$ such that

$$r\pi_{a,y}(x) \leq \varepsilon_{b,z}(x)$$

for all x in $\lambda(E)$, i.e.,

$$(*) \qquad r\left\{\sum_{j=1}^{\infty} \|x_j\|\ |a_j y_j|\right\} \leq \sup_{u \in \mathrm{U}^\circ} \left\{\sum_{j=1}^{\infty} |< x_j, u > b_j z_j|\right\}$$

for all x in $\lambda(E)$. Let $C = (C_j)$ be an element in $\ell^1(E)$. Define $C^n = (C_j^n)$ for every n in $\mathbb{N}$ such that

$$C_j^n = \frac{C_j}{|b_j z_j|} \quad \text{if} \quad j \leq n$$

The details concerning the $\sigma\mu$-topology and the related aspects can be found from [13]. In this paper we show that Dvoretzky-Rogers theorem holds if the traditional normal topology is replaced by $\sigma\mu$-topology.

All classical notations and properties concerning locally convex spaces and sequence spaces are taken from [8] and [14]. We adhere to [10] and [16] for nuclearity and [9] and [16] for tensor products.

Given locally convex spaces E and F, the symbol $E \simeq F$ has the following meaning: E and F are equal as vector spaces and the identity map is a topological isomorphism between them.

Given a perfect AK-sequence space μ, (cf.[14]) a sequence space λ which is μ-perfect and $(E, \|\ \|)$ a normed space, we consider the following generalized sequence spaces (cf. [4] and [11]):

$$\lambda[E] = \{x = (x_n) \in \omega(E) : \{< x_n, u >\}_n \in \lambda, \forall u \in E^*\}$$

provided with the ε-topology generated by the family $\{\varepsilon_{a,y} : a \in \lambda^\mu, y \in \mu^\times\}$ of semi-norms where

$$\varepsilon_{a,y}(x) = \sup_{u \in \mathsf{U}^\circ} pa, y(\{< x_n, u >\}) = \sup_{u \in \mathsf{U}^\circ} \sum_{n=1}^{\infty} | < x_n, u > a_n y_n |,$$

U° being the (absolute) polar set in E^* of the closed unit ball U of E. The subspace of $\lambda[E]$ of all the elements x such that the n-th section $\{x^{(n)}\}$ converges to x for the ε-topology is denoted by $\lambda(E)$ and we consider it endowed with the induced topology. Finally,

$$\lambda\{E\} = \{x = (x_n) \in \omega(E) : \{\|x_n\|\}_n \in \lambda\}$$

endowed with the π-topology defined by the family $\{\pi_{a,y} : a \in \lambda^\mu, y \in \mu^\times\}$ of semi-norms where

$$\pi_{a,y}(x) = p_{a,y}(\|x_n\|) = \sum_{n=1}^{\infty} \|x_n\|\ |a_n y_n|.$$

Recently, it has been investigated (cf. [15]) that

Theorem A. *The space $(\lambda, \sigma\mu)$ is nuclear if and only if $\lambda^\mu \mu^\times = \ell^1 \lambda^\mu \mu^\times$.*

As a direct consequence of this result we obtain the following:

Corollary B. *The space $(\lambda, \sigma\mu)$ is nuclear if and only if $\lambda^\mu \mu^\times = \ell^p \lambda^\mu \mu^\times$ for some* (*each*) $p \geq 1$.

We need the following Lemma of Dvoretzky-Rogers [5].

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