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QUOTIENTS OF GROUPS

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Abstract. Right loop right multiplication groups are identified as groups generated by a right transversal to a corefree subgroup.

1. Introduction

A quotient, or homomorphic image, of a group is the set of all cosets of some normal subgroup, together with the natural binary operation. Would it surprise you to learn that sets of cosets of arbitrary, i.e., not necessarily normal, subgroups are also endowed with a natural groupoid structure? This paper introduces such quotients and describes the central role they play in a hard problem from the theory of groupoids : the classification of multiplication semigroups of groupoids.

2. Basic Definitions

A *groupoid* is a set with a single binary operation, hereafter denoted by juxtaposition. A*semigroup* is a groupoid whose binary operation is associative. A *quasigroup* is a groupoid with inverses, that is, a groupoid such that in $xy = z$, knowledge of any two of x, y and z specifies the third uniquely (y is the right inverse of x with respect to z, x is the left inverse of y with respect to z; when z is a two sided identity element, the language here meshes with the common usage of "inverse"). A *group* is a nonempty groupoid whose binary operation makes it both a semigroup and a quasigroup. These four binary systems can viewed schematically as:

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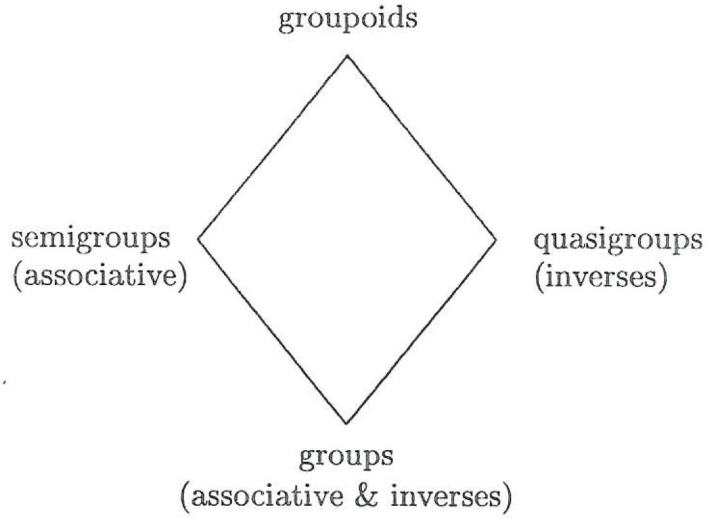

Thus, quasigroups are dual to semigroups in a natural way: quasigroups are "nonassociative groups" while semigroups are "groups without inverses." (For a through overview of groupoids, see [1].)

Given a groupoid Q, for every $q \varepsilon Q$, there corresponds two set maps, *right* and *left translation by* q:

$$R(q) : Q \to Q; x \to xq$$
$$L(q) : Q \to Q; x \to qx.$$

The right translations generate a subsemigroup of the semigroup of all self maps on Q, called the *right multiplication semigroup*, RMlt$Q$, of Q:

$$RMltQ := < R(q) : q \varepsilon Q > .$$

Similarly, the left translations generate the *left multiplication semigroup*, LMlt$Q$, of Q:

$$LMltQ := < L(q) : q \varepsilon Q > .$$

Together, these two semigroups generate the (two-sided) *multiplication semigroup*, Mlt$Q$, of Q:

$$MltQ := < RMltQ, LMltQ > = < R(q), L(q) : q \varepsilon Q > .$$

A groupoid Q is called *right cancellative* if for each q in $Q, R(q)$ is an injective map. If each $R(q)$ is also surjective, we shall say Q is a *right quasigroup*. A *right loop* is a right quasigroup with a left identity element. The analogous left binary systems are defined in the obvious manner.

The determination of which semigroups are Mlt$Q$ for different classes of groupoids Q is a difficult problem. Mlt$Q$ has been characterized for certain groupoids, though [8].

For instance, if Q is a group, then Mlt$Q$ is isomorphic to $(Q \times Q)/\widehat{Z}$, where $\widehat{Z}$ is the image of $Z(Q)$ under the diagonal embedding of Q in $Q \times Q$. A right loop L is an abelian group iff Mlt$L$ is isomorphic to L itself. The multiplication semigroups of various classes of *loops*, i.e., quasigroups with a two-sided identity element, have also been classified [3], [4], [5], but the general classification problem, frought with a welter of thorny obstacles, remains open.

The study of quasigroups and their multiplication semigroups is a rich and active field [2]. Unfortunately, to some, lack of associativity seems to shroud quasigroups in mystery. But nonassociativity is no more mysterious than ordinary subtraction : the integers Z under subtraction form a nonassociative quasigroup. The Cayley table of the binary operation of a finite quasigroup takes the rather appealing form of a *Latin square*, i.e., an $n \times n$ matrix such that each of $\{1, 2, \ldots, n\}$ appears precisely once in each row and precisely once in each column. Conversely, each Latin square determines a unique quasigroup. For example, the Latin square

	1	2	3
1	1	3	2
2	3	2	1
3	2	1	3

determines the unique three element, idempotent quasigroup. An important algebraic reason for interest in quasigroups is the following basic result: Mlt$Q$ is a group if and only if Q is a quasigroup. In this case, Mlt$Q$ is called the *multiplication group*. Similarly, RMlt$Q$ is a group if and only if Q is a right quasigroup.

3. Right Transversals and Right Loops

The quotient groupoid of a group by a subgroup is usually considered only in those instances when the subgroup is normal, in which case the quotient is itself a group. A more general setting arises when the quotient groupoid is a loop [6], [8]. In this case, if the subgroup is Abelian and the transversal is "well-behaved," then the group is actually solvable [4]! In the most general setting the quotient groupoid becomes a right loop. More specifically, the (right) quotient of a group by an arbitrary subgroup can be viewed as a right loop, as will be shown momentarily. This greatly enlarges the scope of the usual investigation of quotients of groups by only normal subgroups.

Recall that a *right transversal* T to a subgroup H of a group G is a complete set of right coset representatives. That is,

$$G = \cup_{t \in T} Ht.$$

There is a natural G action on T, denoted by $*$, given by:

$$t * g := u, \quad \text{where } u \text{ is the representative in } T \text{ of the coset } Htg.$$

This action restricted to T itself, endows T with a binary operation. Thus, T is naturally a groupoid, and can be thought of as the *right quotient of* G by H. We use $\underline{T}$ to denote this groupoid. If H is normal, then $\underline{T}$ is identical with the standard quotient group G/H. In the general case (H not necessarily normal), it is easy to show that $\underline{T}$ is a right loop. It is not necessary that $\underline{T}$ be a loop. For example, let G be the symmetric group on three symbols. That is, let $G =< a, b | a^2 = b^3 = (ab)^2 = 1 >$. Then $T = \{1, b, ba\}$ is a right transversal to $< a >$ in G. Of course, $\underline{T}$ is a right loop. But, the reader can easily check, $\underline{T}$ is not a loop. Thus, right loops appear naturally in the most general setting of right quotients of groups by arbitrary subgroups. Left transversals and left quotients are defined in the obvious manner.

Let T be a right transversal to subgroup H in a group G. There is a natural map $f : T \to \mathrm{RMlt}\underline{T}$, defined by $f(t) = R(t)$. f extends to an epimorphism $F :< T > \to \mathrm{RMlt}\underline{T}$, where $< T >$ is the subgroup generated by T. So RMlt$\underline{T}$ is an image of a subgroup of G. It is easy to show that the kernel of F is the largest normal subgroup of $H \cap < T >$ in $< T >$, the so-called core of $H \cap < T >$ in $< T >$, denoted by core $_{<T>}(H \cap < T >)$. That is:

$$< T > /[core_{<T>}(H \cap < T >)] \cong RMlt\underline{T}.$$

Of course if $\mathrm{core}_{<T>}(H \cap < T >) = 1$, then RMlt$\underline{T}$ is actually a subgroup of G; that is, $< T > \cong \mathrm{RMlt}\underline{T}$. The following lemma summarizes this discussion.

Lemma. *Let T be a right transversal to a subgroup H of a group G and let $K = H \cap < T >$. Then RMlt$\underline{T}$ is a natural homomorphic image of $< T >$. Further, if $core_{<T>}(K) = 1$, then $< T > \cong RMlt\underline{T}$.*

Note. A particularly appealing case of the lemma occurs when $< T >= G$. In this case, if $\mathrm{core}_G(H) = 1$, then G itself is the right multiplication group of some right loop.

Example. Let G be any group. Let H be the trivial subgroup. Then the transversal T is all of G. Obviously both conditions in the theorem are satisfied. And so G becomes the right multiplication group of a right loop. Moreover, this example shows that the lemma offers a generalization of the right regular representation of G.

Example. Let G be a finite simple group. Let H be a subgroup of G of order p, where p is the least prime dividing $|G|$. Let T be any right transversal to H in G. Then $< T >= G$ [7, 1.6.10]. And since $\mathrm{core}_{<T>}(H) = 1$, G is RMlt$\underline{T}$.

Conversely (to the lemma), let L be a right loop. Let H be the subgroup of RMlt$L$ fixing the identity element. The set $T := \{R(x) : x \in L\}$ is a right transversal to H in RMlt$L$. Clearly, $< T >=$RMlt$L$. Moreover, it is easy to show that the core of H in RMlt$L$ is trivial, and that $\underline{T} = L$. These comments combine with the lemma to give the following:

Theorem. *A group G is the right multiplication group of some right loop L if and only if it has a subgroup H and a right transversal T to H in G, such that $\underline{T}$ coincides with L, $core_G(H) = 1$, and $< T >= G$.*

Note. In the theorem, if $\text{core}_G(H)$ is not required to be trivial, then G is the relative multiplication group of some right loop L in another right loop M [6] (relative multiplication groups are special subgroups of multiplication groups [8]).

4. Conclusion

A right quotient of a group by an arbitrary subgroup is equipped with a natural right loop structure. Conversely, every right loop is realizable as such a right quotient. Moreover, right loop right multipilication groups are identified precisely as right quotients of groups generated by a right transversal to an arbitrary subgroup with trivial core. This classification fits in naturally to an on-going research program in the theory of groupoids, and it greatly enlarges the usual study of quotients of groups by only normal subgroups.

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