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AN INVERSE PROBLEM FOR A GENERAL DOUBLY-CONNECTED BOUNDED DOMAIN: AN EXTENSION TO HIGHER DIMENSIONS

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Abstract. The spectral function $\Theta(t) = \sum_{\nu=1}^{\infty} \exp(-t\lambda_\nu)$, where $\{\lambda_\nu\}_{\nu=1}^{\infty}$ are the eigenvalues of the negative Laplacian $-\nabla^2 = -\sum_{i=1}^{3}(\frac{\partial}{\partial x^i})^2$ in the (x^1, x^2, x^3)-space, is studied for an arbitrary doubly connected bounded domain Ω in R^3 together with its smooth inner bounding surface $\tilde{S}_1$ and its smooth outer bounding surface $\tilde{S}_2$, where piecewise smooth impedance boundary conditions on the parts S_1^*, S_2^* of $\tilde{S}_1$ and S_3^*, S_4^* of $\tilde{S}_2$ are considered, such that $\tilde{S}_1 = S_1^* \cup S_2^*$ and $\tilde{S}_2 = S_3^* \cup S_4^*$.

1. Introduction

The underlying inverse problem is to determine some geometric quantities associated with a bounded domain, from a complete knowledge of the eigenvalues $\{\lambda_\nu\}_{\nu=1}^{\infty}$ for the negative Laplacian $-\nabla^2 = -\sum_{i=1}^{3}(\frac{\partial}{\partial x^i})^2$ in the (x^1, x^2, x^3)-space.

Let $\Omega \subseteq R^3$ be a simply connected bounded domain with a smooth bounding surface S. Consider the impedance problem

$$-\nabla^2 u = \lambda u \qquad \text{in } \Omega, \tag{1.1}$$

$$(\frac{\partial}{\partial n}) + \gamma)u = 0 \quad \text{on } S, \tag{1.2}$$

where $\frac{\partial}{\partial n}$ denotes differentiation along the inward pointing normal to S and γ is a positive constant, with $u \in C^2(\Omega) \cap C(\overline{\Omega})$.

Denote its eigenvalues, counted according to multiplicity by

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_\nu \leq \cdots \to \infty \quad \text{as } \nu \to \infty. \tag{1.3}$$

The problem of determining some geometric quantities associated with bounded domain Ω has been discussed by Zayed [6] and Hsu [2] using the asymptotic expansion of the

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spectral function

$$\Theta(t) = \sum_{v=1}^{\infty} \exp(-t\lambda_\nu) \quad \text{as } t \to 0^+. \tag{1.4}$$

The problem (1.1) - (1.2) has been investigated by many authors (see, for example the articles [1 - 5]) in the following special cases:
Case 1. $\gamma = 0$ (the Neumann problem)

$$\Theta(t) = \frac{V}{(4\pi t)^{3/2}} + \frac{|S|}{16\pi t} + \frac{1}{12\pi^{3/2}t^{1/2}} \int_S H dS + a_0 + O(t^{1/2}) \text{ as } t \to 0^+. \tag{1.5}$$

Case 2. $\gamma \to \infty$ (the Dirichlet problem)

$$\Theta(t) = \frac{V}{(4\pi t)^{3/2}} - \frac{|S|}{16\pi t} + \frac{1}{12\pi^{3/2}t^{1/2}} \int_S H dS + a_0 + O(t^{1/2}) \text{ as } t \to 0^+. \tag{1.6}$$

In these formulae, V and $|S|$ are respectively the volume and the surface area of Ω, while $H = \frac{1}{2}(\frac{1}{R_1} + \frac{1}{R_2})$ is the mean curvature of S, where R_1 and R_2 are the principal radii of curvature. It has been shown that the constant term a_0 has the following forms:

$$a_0 = \begin{cases} \frac{7}{512\pi} \int_S (\frac{1}{R_1} - \frac{1}{R_2})^2 dS, \text{ In the case 1 (see [2]),} \\ \frac{1}{512\pi} \int_S (\frac{1}{R_1} - \frac{1}{R_2})^2 dS, \text{ In the case 2 (see [5]),} \end{cases} \tag{1.7}$$

In terms of the mean curvature H and the Gaussian curvature $N = \frac{1}{R_1 R_2}$, the constant term a_0 can be rewritten in the forms:

$$a_0 = \begin{cases} \frac{7}{128\pi} \int_S (H^2 - N) dS, \text{ in the case 1,} \\ \frac{1}{128\pi} \int_S (H^2 - N) dS, \text{ in the case 2.} \end{cases} \tag{1.8}$$

Case 3. (the mixed problem)

If $|S_1|$ is the length of a part S_1 of the bounding surface S with the Neumann boundary condition, and if $|S_2|$ is the length of the remaining part $S_2 = S \backslash S_1$ of S with the Dirichlet boundary condition, such that $S = S_1 \cup S_2$, then with reference to [8,10], we get

$$\Theta(t) = \frac{V}{(4\pi t)^{3/2}} + \frac{|S_1| - |S_2|}{16\pi t} + \frac{1}{12\pi^{3/2}t^{1/2}} \left\{ \int_{S_1} H dS_1 + \int_{S_2} H dS_2 \right\} +$$
$$+ \frac{1}{128\pi} \left\{ 7 \int_{S_1} (H^2 - N) dS_1 + \int_{S_2} (H^2 - N) dS_2 \right\} + O(t^{1/2}) \text{ as } t \to 0^+ \tag{1.9}$$

Zayed [10] has recently discussed the equation (1.1) together with the piecewise smooth impedance boundary conditions:

$$(\frac{\partial}{\partial n_1} + \gamma_1) u = 0 \quad \text{in} \quad S_1, \quad (\frac{\partial}{\partial n_2} + \gamma_2) u = 0 \quad \text{on} \quad S_2, \tag{1.10}$$

where $\frac{\partial}{\partial n_1}$ and $\frac{\partial}{\partial n_2}$ denote differentiations along the inward pointing normals to S_1 and S_2 respectively, in which S_1 is a part of S and $S_2 = S \setminus S_1$ is the remaining part of S, such that $S = S_1 \cup S_2$, while the impedances γ_1 and γ_2 are positive constants. The author has calculated only the first three terms of the asymptotics of the heat kernel of the problem (1.1), (1.10) and has determined some geometric quantities of the domain Ω.

Now, let Ω be an arbitrary doubly connected domain in R^3 consisting of a simply connected bounded inner domain Ω_1 with a smooth bounding surface $\tilde{S}_1$ and a simply connected bounded outer domain $\Omega_2 \supset \overline{\Omega}_1$ with a smooth bounding surface $\tilde{S}_2$ where $\overline{\Omega}_1 = \Omega_1 \cup \tilde{S}_1$. Suppose that the eigenvalues (1.3) are given for the eigenvalue equation

$$-\nabla^2 u = \lambda u \qquad \text{in } \Omega, \tag{1.11}$$

together with the impedance boundary conditions

$$(\frac{\partial}{\partial n_1} + \gamma_1)u = 0 \quad \text{on} \quad \tilde{S}_1, \quad (\frac{\partial}{\partial n_2} + \gamma_2)u = 0 \quad \text{on} \quad \tilde{S}_2, \tag{1.12}$$

where γ_1 and γ_2 are positive constants.

Zayed [7] has recently discussed the problem (1.11), (1.12) and has determined the first four terms of the asymptotic expansions of the spectral function $\Theta(t)$ for small positive t. The author has determined some geometric quantities associated with the problem (1.11)-(1.12). The object of this paper is to discuss a more general inverse eigenvalue problem consisting of the eigenvalue equation (1.11) together with the piecewise smooth impedance boundary conditions:

$$(\frac{\partial}{\partial n_j} + \gamma_j)u = 0 \quad \text{on} \quad S_j^* \quad (j = 1, 2, 3, 4), \tag{1.13}$$

where S_1^* is a part of the inner bounding surface $\tilde{S}_1$ of Ω and $S_2^* = \tilde{S}_1 \setminus S_1^*$ is the remaining part of $\tilde{S}_1$, such that $\tilde{S}_1 = S_1^* \cup S_2^*$, while S_3^* is a part of the outer bounding surface $\tilde{S}_2$ of Ω and $S_4^* = \tilde{S}_2 \setminus S_3^*$ is the remaining part of $\tilde{S}_2$, such that $\tilde{S}_2 = S_3^* \cup S_4^*$, and the impedances γ_j $(j = 1, 2, 3, 4)$ are positive constants.

The basic problem is to determine some geometric quantities associated with the arbitrary doubly connected domain Ω in R^3 from the complete knowledge of the eigenvalues $\{\lambda_\nu\}_{\nu=1}^{\infty}$ for the impedance problem (1.11), (1.13) using the asymptotic expansion of $\Theta(t)$ as $t \to 0^+$.

We close this section with the remark that a similar inverse problem has been discussed recently by Zayed [12] where Ω is a two-dimensional doubly connected bounded domain.

2. Statement of the Results

Suppose that the outer bounding surface $\tilde{S}_2$ of the domain Ω is given locally by infinitely differentiable functions $x^\beta = y^\beta(\sigma_2)(\beta = 1, 2, 3)$ of the parameters σ_2^i $(i = 1, 2)$.

If these parameters are chosen so that σ_2^i = constant, are lines of curvature, the first and second fundamental forms of $\tilde{S}_2$ can be written in the form:

$$II_1(\sigma_2, \Delta\sigma_2) = \sum_{i=1}^{2} g_{ii}(\sigma_2)(\Delta\sigma_2^i)^2, \tag{2.1}$$

and

$$II_2(\sigma_2, \Delta\sigma_2) = \sum_{i=1}^{2} d_{ii}(\sigma_2)(\Delta\sigma_2^i)^2, \tag{2.2}$$

In terms of the coefficients g_{ii}, d_{ii} the principal radii of curvature are $R_{ii} = g_{ii}/d_{ii}$ $(i = 1, 2)$. Consequently, the mean curvature H_1 and the Gaussian curvature N_1 of $\tilde{S}_2$ are given as follows:

$$H_1 = \frac{1}{2}(\frac{1}{R_{11}} + \frac{1}{R_{22}}) \quad \text{and } N_1 = \frac{1}{R_{11}R_{22}}.$$

Similarly, suppose that the inner bounding surface $\tilde{S}_1$ of Ω is given locally by infinitely differentiable functions $x^\beta = y^\beta(\sigma_1)$ $(\beta = 1, 2, 3)$ of the parameters $\sigma_1^i (i = 1, 2)$. If these parameters are chosen so that σ_1^i = constant, are lines of curvature, the first and second fundamental forms of $\tilde{S}_1$ can be written in the forms:

$$II_1^*(\sigma_1, \Delta\sigma_1) = \sum_{i=1}^{2} g_{ii}^*(\sigma_1)(\Delta\sigma_1^i)^2, \tag{2.3}$$

and

$$II_2^*(\sigma_1, \Delta\sigma_) = \sum_{i=1}^{2} d_{ii}^*(\sigma_1)(\Delta\sigma_1^i)^2, \tag{2.4}$$

In terms of the coefficients g_{ii}^* ,d_{ii}^* the principal radii of curvature are $R_{ii}^* = g_{ii}^*/d_{ii}^* (i = 1, 2)$. Consequently, the mean curvature H_1^* and the Gaussian curvature N_1^* of $\tilde{S}_1$ are given as follows:

$$H_1^* = \frac{1}{2}(\frac{1}{R_{11}^*} + \frac{1}{R_{22}^*}) \quad \text{and } N_1^* = \frac{1}{R_{11}^* R_{22}^*}.$$

Let $|S_1^*|$, $|S_2^*|$ be the surface areas of the parts S_1^*, S_2^* of $\tilde{S}_1$ respectively, and let $|S_3^*|$, $|S_4^*|$ be the surface areas of the parts S_3^*, S_4^* of $\tilde{S}_2$ respectively. Then, the results of our main problem (1.11)-(1.13) can be summarized in the following cases:

Case 1. $0 < \gamma_1 << 1$, $\gamma_2 >> 1$, $0 < \gamma_3 << 1$, $\gamma_4 >> 1$)

$$\Theta(t) = \frac{V}{(4\pi t)^{3/2}} + \frac{1}{16\pi t}\Big\{[|S_1^*| - (|S_2^*| - 2\gamma_2^{-1}\int_{S_2^*} H_1^* dS_2^*)] +$$
$$+[|S_3^*| - (|S_4^*| - 2\gamma_4^{-1}\int_{S_4^*} H_1 dS_4^*)]\Big\} +$$
$$+\frac{1}{12\pi^{3/2}t^{1/2}}\Big\{\int_{S_1^*}(H_1^* - 3\gamma_1)dS_1^* + \int_{S_2^*} H_1^* dS_2^* +$$

$$
\begin{aligned}
&+\int_{S_3^*}(H_1-3\gamma_3)dS_3^*+\int_{S_4^*}H_1dS_4^*\Big\}\\
&+\frac{1}{128\pi}\Big\{7\int_{S_1^*}[(H_1^*-3\gamma_1)^2-(N_1^*-\frac{26}{7}\gamma_1H_1^*+\frac{47}{7}\gamma_1^2)]dS_1^*\\
&+\int_{S_2^*}[H_1^{*^2}-(N_1^*-16\gamma_2^{-1}H_1^*)]dS_2^*\\
&+7\int_{S_3^*}[(H_1-3\gamma_3)^2-(N_1-\frac{26}{7}\gamma_3H_1+\frac{47}{7}\gamma_3^2)]dS_3^*\\
&+\int_{S_4^*}[H_1^2-(N_1-16\gamma_4^{-1}H_1)]dS_4^*\Big\}\\
&+(\frac{t}{\pi^3})^{1/2}\Big\{\frac{13}{1440}\int_{S_1^*}(H_1^*-3\gamma_1)^3dS_1^*-\frac{1}{315}\int_{S_2^*}H_1^{*^3}dS_2^*\\
&+\frac{13}{1440}\int_{S_3^*}(H_1-3\gamma_3)^3dS_3^*-\frac{1}{315}\int_{S_4^*}H_1^3dS_4^*\Big\}\\
&+O(t)\quad \text{as } t\to 0^+. \qquad (2.5)
\end{aligned}
$$

Case 2. $(0<\gamma_1<<1,\ \gamma_2>>1,\ \gamma_3>>1,\ 0<\gamma_4<<1)$

In this case, the asymptotic expansion of $\Theta(t)$ has the same form (2.5) with the interchanges $\gamma_3\leftrightarrow\gamma_4$, $S_3^*\leftrightarrow S_4^*$.

Case 3. $(\gamma_1,\ \gamma_2>>1,\ 0<\gamma_3,\ \gamma_4<<1)$

$$
\begin{aligned}
\Theta(t)=&\frac{V}{(4\pi t)^{3/2}}+\frac{1}{16\pi t}\Big\{\sum_{i=3}^{4}|S_i^*|-\sum_{i=1}^{2}(|S_i^*|-2\gamma_i^{-1}\int_{S_i^*}H_1^*dS_i^*)\Big\}\\
&+\frac{1}{12\pi^{3/2}t^{1/2}}\Big\{\sum_{i=1}^{2}\int_{S_i^*}H_1^*dS_i+\sum_{i=3}^{4}\int_{S_i^*}(H_1-3\gamma_i)dS_i^*\Big\}\\
&+\frac{1}{128\pi}\Big\{\sum_{i=1}^{2}\int_{S_i^*}[H_1^{*^2}-(N_1^*-16\gamma_i^{-1}H_1^*)]dS_i^*\\
&+7\sum_{i=3}^{4}\int_{S_i^*}[(H_1-3\gamma_i)^2-(N_1-\frac{26}{7}\gamma_iH_1+\frac{47}{7}\gamma_i^2)]dS_i^*\Big\}\\
&+(\frac{t}{\pi^3})^{1/2}\Big\{-\frac{1}{315}\sum_{i=1}^{2}\int_{S_1^*}H_1^{*^3}dS_i^*+\frac{13}{1440}\sum_{i=3}^{4}\int_{S_i^*}(H_1-3\gamma_i)^3dS_i^*\Big\}\\
&+O(t)\quad \text{as } t\to 0^+. \qquad (2.6)
\end{aligned}
$$

Case 4. $(0<\gamma_1,\ \gamma_2<<1,\ \gamma_3,\ \gamma_4>>1)$

In this case, the asymptotic expansion of $\Theta(t)$ has the same form (2.6) with the interchanges $\gamma_1\leftrightarrow\gamma_3$, $\gamma_2\leftrightarrow\gamma_4$, $H_1\leftrightarrow H_1^*$ and $S_i^*(i=1,2)\leftrightarrow S_i^*\ (i=3,4)$.

Case 5. $(\gamma_1>>1,\ 0<\gamma_2<<1,\ \gamma_3>>1,\ 0<\gamma_4<<1)$

In this case, the asymptotic expansion of $\Theta(t)$ as $t \to 0$ has the same form (2.5) with the interchanges $\gamma_1 \leftrightarrow \gamma_2$, $\gamma_3 \leftrightarrow \gamma_4$,$S_1^* \leftrightarrow S_2^*$ and $S_3^* \leftrightarrow S_4^*$.

Case 6. $(\gamma_1 >> 1, 0 < \gamma_2 << 1, 0 < \gamma_3 << 1, \gamma_4 >> 1)$

In this case, the asymptotic expansion of $\Theta(t)$ as $t \to 0$ has the same form (2.5) with the interchanges $\gamma_1 \leftrightarrow \gamma_2$ and $S_1^* \leftrightarrow S_2^*$.

Case 7. $(0 < \gamma_1 << 1, \gamma_2 >> 1, \gamma_3, \gamma_4 >> 1)$

$$\begin{aligned}\Theta(t) = & \frac{V}{(4\pi t)^{3/2}} + \frac{1}{16\pi t}\Big\{[|S_1^*| - (|S_2^*| - 2\gamma_2^{-1}\int_{S_2^*} H_1^* dS_2^*)] \\ & - \sum_{i=3}^{4}(|S_1^*| - 2\gamma_i^{-1}\int_{S_i^*} H_1 dS_i^*)\Big\} \\ & + \frac{1}{12\pi^{3/2}t^{1/2}}\Big\{\int_{S_1^*}(H_1^* - 3\gamma_1)dS_1^* + \int_{S_2^*} H_1^* dS_2^* + \sum_{i=3}^{4}\int_{S_i^*} H_1 dS_i^*\Big\} \\ & + \frac{1}{128\pi}\Big\{7\int_{S_1^*}[(H_1^* - 3\gamma_1)^2 - (N_1^* - \frac{26}{7}\gamma_1 H_1^* + \frac{47}{7}\gamma_1^2)]dS_1^* \\ & + \int_{S_2^*}[H_1^{*^2} - (N_1^* - 16\gamma_2^{-1}H_1^*)]dS_2^* \\ & + \sum_{i=3}^{4}\int_{S_1^*}[(H_1^2 - (N_1 - 16\gamma_i^{-1}H_1)]dS_i^*\Big\} \\ & + (\frac{t}{\pi^3})^{1/2}\Big\{\frac{13}{1440}\int_{S_1^*}(H_1^* - 3\gamma_1)^3 dS_1^* - \frac{1}{315}\int_{S_2^*} H_1^{*^3} dS_2^* \\ & - \frac{1}{315}\sum_{i=3}^{4}\int_{S_i^*} H_1^3 dS_i^*\Big\} + O(t) \quad \text{as } t \to 0^+. \end{aligned} \tag{2.7}$$

Case 8. $(\gamma_1 >> 1, 0 < \gamma_2 << 1, \gamma_3, \gamma_4 >> 1)$

In this case, the asymptotic expansion of $\Theta(t)$ as $t \to 0$ has the same form (2.7) with the interchanges $\gamma_1 \leftrightarrow \gamma_2$ and $S_1^* \leftrightarrow S_2^*$.

Case 9. $(\gamma_1, \gamma_2 >> 1, 0 < \gamma_3 << 1, \gamma_4 >> 1)$

In this case, the asymptotic expansion of $\Theta(t)$ as $t \to 0$ has the same form (2.7) with the interchanges $\gamma_1 \leftrightarrow \gamma_3$, $\gamma_2 \leftrightarrow \gamma_4$, $S_1^* \leftrightarrow S_3^*$, $S_2^* \leftrightarrow S_4^*$ and $H_1 \leftrightarrow H_1^*$.

Case 10. $(\gamma_1, \gamma_2 >> 1, \gamma_3 >> 1, 0 < \gamma_4 << 1)$

In this case, the asymptotic expansion of $\Theta(t)$ as $t \to 0$ has the same form (2.7) with the interchanges $\gamma_1 \leftrightarrow \gamma_4$, $\gamma_2 \leftrightarrow \gamma_3$, $S_1^* \leftrightarrow S_4^*$, $S_2^* \leftrightarrow S_3^*$ and $H_1 \leftrightarrow H_1^*$.

Case 11. $(\gamma_1 >> 1, 0 < \gamma_2 << 1, 0 < \gamma_3, \gamma_4 << 1)$

$$\Theta(t) = \frac{V}{(4\pi t)^{3/2}} + \frac{1}{16\pi t}\Big\{[|S_2^*| - (|S_1^*| - 2\gamma_1^{-1}\int_{S_1^*} H_1^* dS_2^*)] + \sum_{i=3}^{4}(|S_i^*|\Big\}$$

$$+\frac{1}{12\pi^{3/2}t^{1/2}}\Big\{\int_{S_1^*} H_1^* dS_1^* + \int_{S_2^*}(H_1^* - 3\gamma_2)dS_2^* + \sum_{i=3}^{4}\int_{S_i^*}(H_1 - 3\gamma_1)dS_i^*\Big\}$$

$$+\frac{1}{128\pi}\Big\{\int_{S_1^*}[H_1^{*^2} - (N_1^* - 16\gamma_1^{-1}H_1^*)]dS_1^*$$

$$+7\int_{S_2^*}[H_1^* - 3\gamma_2)^2 - (N_1^* - \frac{26}{7}\gamma_2 H_1^* + \frac{47}{7}\gamma_2^2)]dS_2^*$$

$$+7\sum_{i=3}^{4}\int_{S_i^*}[(H_1 3\gamma_i)^2 - (N_1 - \frac{26}{7}\gamma_i H_1 + \frac{47}{7}\gamma_i^2)]dS_i^*\Big\}$$

$$+(\frac{t}{\pi^3})^{1/2}\Big\{-\frac{1}{315}\int_{S_1^*} H_1^{*^3} dS_1^* + \frac{13}{1440}\int_{S_2^*}(H_1^* - 3\gamma_2)^3 dS_2^*$$

$$+\frac{13}{1440}\sum_{i=3}^{4}\int_{S_i^*}(H_1 - 3\gamma_i)^3 dS_i^*\Big\} + O(t) \quad \text{as } t \to 0^+. \tag{2.8}$$

Case 12. $(0 < \gamma_1 << 1, \gamma_2 >> 1, 0 < \gamma_3, \gamma_4 << 1)$

In this case, the asymptotic expansion of $\Theta(t)$ as $t \to 0$ has the same form (2.8) with the interchanges $\gamma_1 \leftrightarrow \gamma_2$ and $S_1^* \leftrightarrow S_2^*$.

Case 13. $(0 < \gamma_1, \gamma_2 << 1, \gamma_3 >> 1, 0 < \gamma_4 << 1)$

In this case, the asymptotic expansion of $\Theta(t)$ as $t \to 0$ has the same form (2.8) with the interchanges $\gamma_1 \leftrightarrow \gamma_3$, $\gamma_2 \leftrightarrow \gamma_4$, $S_1^* \leftrightarrow S_3^*$, $S_2^* \leftrightarrow S_4^*$ and $H_1^* \leftrightarrow H_1$.

Case 14. $(0 < \gamma_1, \gamma_2 << 1, 0 < \gamma_3 << 1, \gamma_4 >> 1)$

In this case, the asymptotic expansion of $\Theta(t)$ as $t \to 0$ has the same form (2.8) with the interchanges $\gamma_1 \leftrightarrow \gamma_4$, $\gamma_2 \leftrightarrow \gamma_3$, $S_1^* \leftrightarrow S_4^*$, $S_2^* \leftrightarrow S_3^*$ and $H_1^* \leftrightarrow H_1$.

Case 15. $(0 < \gamma_1, \gamma_2, \gamma_3, \gamma_4 << 1)$

$$\Theta(t) = \frac{V}{(4\pi t)^{3/2}} + (\sum_{i=1}^{4}|S_i^*|)/16\pi t$$

$$+\frac{1}{12\pi^{3/2}t^{1/2}}\Big\{\sum_{i=1}^{2}\int_{S_1^*}(H_1^* - 3\gamma_i)dS_i^* + \sum_{i=3}^{4}\int_{S_i^*}(H_1 - 3\gamma_i)dS_i^*\Big\}$$

$$+\frac{7}{128\pi}\Big\{\sum_{i=1}^{2}\int_{S_1^*}[(H_1^* - 3\gamma_i)^2 - (N_1^* - \frac{26}{7}\gamma_i H_1^* + \frac{47}{7}\gamma_i^2)]dS_i^*$$

$$+\sum_{i=3}^{4}\int_{S_i^*}[(H_1 - 3\gamma_i)^2 - (N_1 - \frac{26}{7}\gamma_i H_1 + \frac{47}{7}\gamma_i^2)]dS_i^*\Big\}$$

$$+\frac{13}{1440}(\frac{t}{\pi^3})^{1/2}\Big\{\sum_{i=1}^{2}\int_{S_1^*}(H_1^* - 3\gamma_i)^3 dS_i^* + \sum_{i=3}^{4}\int_{S_i^*}(H_1 - 3\gamma_i)^3 dS_i^*\Big\}$$

$$+O(t) \quad \text{as } t \to 0^+. \tag{2.9}$$

Case 16. $(\gamma_1, \gamma_2, \gamma_3, \gamma_4 >> 1)$

$$\Theta(t) = \frac{V}{(4\pi t)^{3/2}} - \frac{1}{16\pi t}\Big\{\sum_{i=1}^{2}(|S_i^*| - 2\gamma_i^{-1}\int_{S_i^*} H_1^* dS_i^*)$$
$$+\sum_{i=3}^{4}(|S_i^*| - 2\gamma_i^{-1}\int_{S_i^*} H_1 dS_i^*)\Big\}$$
$$+\frac{1}{12\pi^{3/2}t^{1/2}}\Big\{\sum_{i=1}^{2}\int_{S_1^*} H_1^* dS_i^* + \sum_{i=3}^{4}\int_{S_i^*} H_1 dS_i^*\Big\}$$
$$+\frac{1}{128\pi}\Big\{\sum_{i=1}^{2}\int_{S_1^*}[H_1^{*2} - (N_1^* - 16\gamma_i^{-1}H_1^*)]dS_i^*$$
$$+\sum_{i=3}^{4}\int_{S_i^*}[H_1^2 - (N_1 - 16\gamma_i^{-1}H_1)]dS_i^*\Big\}$$
$$-\frac{1}{315}(\frac{t}{\pi^3})^{1/2}\Big\{\sum_{i=1}^{2}\int_{S_i^*} H_1^{*3} dS_i^* + \sum_{i=3}^{4}\int_{S_i^*} H_1^3 dS_i^*\Big\}$$
$$+O(t) \quad \text{as } t \to 0^+. \tag{2.10}$$

With reference to the formulae (1.5) - (1.9), and to the articles [7], [8], [10], [11], the asymptotic expansions (2.5)-(2.10) may be interpreted as follows:
(i) Ω is an arbitrary doubly connected domain in R^3 and we have the piecewise impedance boundary conditions (1.13) with small/large impedances γ_j $(j = 1,2,3,4)$ as indicated in the specifications of the sixteen respective cases.
(ii) For the first five terms, Ω is an arbitrary doubly connected bounded domain in R^3 of volume V.

In (2.5), the part S_1^* of $\tilde{S}_1$ is of surface are $|S_1^*|$, mean curvature $(H_1^* - 3\gamma_1)$ and Gaussian curvature $(N_1^* - \frac{26}{7}\gamma_1 H_1^* + \frac{47}{7}\gamma_1^2)$ together with the Neumann boundary conditions, while the remaining par $S_2^* = \tilde{S}_1 \setminus S_1^*$ of $\tilde{S}_1$ is of surface area $(|S_2^*| - 2\gamma_2^{-1}\int_{S_2^*} H_1^* dS_2^*)$, mean curvature H_1^* and Gaussian curvature $(N_1^* - 16\gamma_2^{-1}H_1^*)$ together with the Dirichlet boundary condition. Similarly, the part S_3^* of $\tilde{S}_2$ is of surface area $|S_3^*|$, mean curvature $(H_1 - 3\gamma_3)$ and Gaussian curvature $(N_1 - \frac{26}{7}\gamma_3 H_1 + \frac{47}{7}\gamma_3^2)$ together with the Neumann boundary conditions, while the remaining part $S_4^* = \tilde{S}_2 \setminus S_3^*$ of $\tilde{S}_2$ is of surface area $(|S_4^*| - 2\gamma_4^{-1}\int_{S_4^*} H_1 dS_4^*)$, mean curvature H_1 and Gaussian curvature $(N_1 - 16\gamma_4^{-1}H_1)$ together with the Dirichlet boundary condition.

Similarly, we can interpret the asymptotic expansions (2.6)-(2.10) in a similar way as we have done above for the formula (2.5).

3. Formulation of the Mathematical Problem

In analogy with the two-dimensional problem [12], it is easy to show that the spectral function $\Theta(t)$ associated with the problem (1.11)-(1.13) is given by the formula:

$$\Theta(t) = \iiint_{\Omega} G(\underset{\sim}{x}\underset{\sim}{x};t)d\underset{\sim}{x} \tag{3.1}$$

where $G(\underset{\sim}{x}_1\underset{\sim}{x}_2;t)$ is the Green's function for the heat equation

$$\nabla^2 u = \frac{\partial u}{\partial t}, \tag{3.2}$$

subject to the piecewise smooth impedance boundary conditions

$$(\frac{\partial}{\partial n_j} + \gamma_j)G(\underset{\sim}{x}_1, \underset{\sim}{x}_2;t) = 0 \quad \text{for } \underset{\sim}{x}_1 \in S_j^*, \quad (j = 1,2,3,4), \tag{3.3}$$

and the initial condition

$$\lim_{t\to 0+} G(\underset{\sim}{x}_1, \underset{\sim}{x}_2;t) = \delta(\underset{\sim}{x}_1 - \underset{\sim}{x}_2), \tag{3.4}$$

where $\delta(\underset{\sim}{x}_1 - \underset{\sim}{x}_2)$ is the Dirac delta function located at the source point $\underset{\sim}{x}_2$. Let us write

$$G(\underset{\sim}{x}_1, \underset{\sim}{x}_2;t) = G_0(\underset{\sim}{x}_1, \underset{\sim}{x}_2;t) + \chi(\underset{\sim}{x}_1, \underset{\sim}{x}_2;t), \tag{3.5}$$

where

$$G_0(\underset{\sim}{x}_1, \underset{\sim}{x}_2;t) = (4\pi t)^{-3/2} \exp\left\{ -\frac{|\underset{\sim}{x}_1 - \underset{\sim}{x}_2|^2}{4t} \right\}, \tag{3.6}$$

is the "fundamental solution" of the heat equation (3.2), while $\chi(\underset{\sim}{x}_1, \underset{\sim}{x}_2;t)$ is the "regular solution" chosen in such a way that $G(\underset{\sim}{x}_1, \underset{\sim}{x}_2;t)$ satisfies the piecewise smooth impedance boundary conditions (3.3).

On setting $\underset{\sim}{x}_1 = \underset{\sim}{x}_2 = \underset{\sim}{x}$, we find that

$$\Theta(t) = \frac{V}{(4\pi t)^{3/2}} + R(t), \tag{3.7}$$

where

$$R(t) = \iiint_{\Omega} \chi(\underset{\sim}{x}, \underset{\sim}{x};t)d\underset{\sim}{x}. \tag{3.8}$$

The problem now is to determine the asymptotic expansion of $R(t)$ as $t \to 0^+$. In what follows, we shall use Laplace transforms with respect to t, and use s^2 as the Laplace transform parameter; thus we define

$$\overline{G}(\underset{\sim}{x}_1, \underset{\sim}{x}_2;s^2) = \int_0^{+\infty} e^{-s^2 t} G(\underset{\sim}{x}_1, \underset{\sim}{x}_2;t)dt. \tag{3.9}$$

An application of the Laplace transform to the heat equation (3.2) shows that $\overline{G}(\underset{\sim}{x}_1, \underset{\sim}{x}_2;s^2)$ satisfies the membrane equation

$$(\nabla^2 - s^2)\overline{G}(\underset{\sim}{x}_1, \underset{\sim}{x}_2;s^2) = -\delta(\underset{\sim}{x}_1, \underset{\sim}{x}_2) \text{ in } \Omega, \tag{3.10}$$

together with the piecewise smooth impedance boundary conditions

$$(\frac{\partial}{\partial n_j} + \gamma_j)\overline{G}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2) = 0 \quad \text{for } \underset{\sim}{x}_1 \in S_j^*, (j = 1,2,3,4). \tag{3.11}$$

The asymptotic expansion of $R(t)$ as $t \to 0^+$, may then be deduced directly from the asymptotic expansion of $\overline{R}(s^2)$ as $s \to \infty$, where

$$\overline{R}(s^2) = \int\int\int_{\Omega} \overline{\chi}(\underset{\sim}{x}, \underset{\sim}{x}; s^2) d\underset{\sim}{x}. \tag{3.12}$$

4. Construction of the Green's Function

It is well known (see [5], [6]) that the membrane equation (3.10) has the fundamental solution

$$\overline{G}_0(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2) = \frac{\exp(-s r_{\underset{\sim}{x}_1 \underset{\sim}{x}_2})}{4\pi r_{\underset{\sim}{x}_1 \underset{\sim}{x}_2}}, \tag{4.1}$$

where $r_{\underset{\sim}{x}_1 \underset{\sim}{x}_2} = |\underset{\sim}{x}_1 - \underset{\sim}{x}_2|$ is the distance between the points $\underset{\sim}{x}_1 = (x_1^1, x_1^2, x_1^3)$ and $\underset{\sim}{x}_2 = (x_2^1, x_2^2, x_2^3)$ of the region Ω. The existence of this solution enables us to construct integral equations for $\overline{G}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2)$ satisfying the piecewise smooth impedance boundary conditions (3.11) for small/large impedances $\gamma_j (j = 1,2,3,4)$ as indicated in the specifications of the sixteen respective cases. Therefore, Green's theorem gives:

Case 1. $(0 < \gamma_1 << 1.\ \gamma_2 >> 1, 0 < \gamma_3 << 1, \gamma_4 >> 1)$

In this case, we have the integral equation

$$\begin{aligned}
\overline{G}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2) = & \frac{\exp(-s r_{\underset{\sim}{x}_1 \underset{\sim}{x}_2})}{4\pi r_{\underset{\sim}{x}_1 \underset{\sim}{x}_2}} \\
& + \frac{1}{2\pi} \int_{S_1^*} \overline{G}(\underset{\sim}{x}_1, \underset{\sim}{y}; s^2) \Big\{ \frac{\partial}{\partial n_{1\underset{\sim}{y}}} [\frac{\exp(-s r_{\underset{\sim}{y}\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}\underset{\sim}{x}_2}}] + \gamma_1 \frac{\exp(-s r_{\underset{\sim}{y}\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}\underset{\sim}{x}_2}} \Big\} d\underset{\sim}{y} \\
& - \frac{1}{2\pi} \int_{S_2^*} \frac{\partial}{\partial n_{2\underset{\sim}{y}}} \overline{G}(\underset{\sim}{x}_1, \underset{\sim}{y}; s^2) \Big\{ \frac{\exp(-s r_{\underset{\sim}{y}\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}\underset{\sim}{x}_2}} + \gamma_2^{-1} \frac{\partial}{\partial n_{2\underset{\sim}{y}}} [\frac{\exp(-s r_{\underset{\sim}{y}\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}\underset{\sim}{x}_2}}] \Big\} d\underset{\sim}{y} \\
& - \frac{1}{2\pi} \int_{S_3^*} \overline{G}(\underset{\sim}{x}_1, \underset{\sim}{y}; s^2) \Big\{ \frac{\partial}{\partial n_{3\underset{\sim}{y}}} [\frac{\exp(-s r_{\underset{\sim}{y}\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}\underset{\sim}{x}_2}}] + \gamma_3 \frac{\exp(-s r_{\underset{\sim}{y}\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}\underset{\sim}{x}_2}} \Big\} d\underset{\sim}{y} \\
& + \frac{1}{2\pi} \int_{S_4^*} \frac{\partial}{\partial n_{4\underset{\sim}{y}}} \overline{G}(\underset{\sim}{x}_1, \underset{\sim}{y}; s^2) \Big\{ \frac{\exp(-s r_{\underset{\sim}{y}\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}\underset{\sim}{x}_2}} + \gamma_4^{-1} \frac{\partial}{\partial n_{4\underset{\sim}{y}}} [\frac{\exp(-s r_{\underset{\sim}{y}\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}\underset{\sim}{x}_2}}] \Big\} d\underset{\sim}{y}.
\end{aligned} \tag{4.2}$$

Similarly, the integral equations of $\overline{G}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2)$ for the other fifteen respective cases can be found easily.

On applying the iteration methods (see [6], [9], [10]) to the integral equation (4.2), we obtain the Green's function $\overline{G}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2)$ which has the regular part:

$$
\begin{aligned}
&\overline{\chi}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2) \\
&= \frac{1}{8\pi^2}\int_{S_1^*} \frac{\exp(-sr_{\underset{\sim}{x}_1\underset{\sim}{y}})}{r_{\underset{\sim}{x}_1\underset{\sim}{y}}}\Big\{\frac{\partial}{\partial n_{1\underset{\sim}{y}}}\Big[\frac{\exp(-sr_{\underset{\sim}{y}\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}\underset{\sim}{x}_2}}\Big] + \gamma_1 \frac{\exp(-sr_{\underset{\sim}{y}\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}\underset{\sim}{x}_2}}\Big\}d\underset{\sim}{y} \\
&-\frac{1}{8\pi^2}\int_{S_2^*} \frac{\partial}{\partial n_{2\underset{\sim}{y}}}\Big[\frac{\exp(-sr_{\underset{\sim}{x}_1\underset{\sim}{y}})}{r_{\underset{\sim}{x}_1\underset{\sim}{y}}}\Big]\Big\{\frac{\exp(-sr_{\underset{\sim}{y}\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}\underset{\sim}{x}_2}} + \gamma_2^{-1}\frac{\partial}{\partial n_{2\underset{\sim}{y}}}\Big[\frac{\exp(-sr_{\underset{\sim}{y}\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}\underset{\sim}{x}_2}}\Big]\Big\}d\underset{\sim}{y} \\
&-\frac{1}{8\pi^2}\int_{S_3^*} \frac{\exp(-sr_{\underset{\sim}{x}_1\underset{\sim}{y}})}{r_{\underset{\sim}{x}_1\underset{\sim}{y}}}\Big\{\frac{\partial}{\partial n_{3\underset{\sim}{y}}}\Big[\frac{\exp(-sr_{\underset{\sim}{y}\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}\underset{\sim}{x}_2}}\Big] + \gamma_3 \frac{\exp(-sr_{\underset{\sim}{y}\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}\underset{\sim}{x}_2}}\Big\}d\underset{\sim}{y} \\
&+\frac{1}{8\pi^2}\int_{S_4^*} \frac{\partial}{\partial n_{4\underset{\sim}{y}}}\Big[\frac{\exp(-sr_{\underset{\sim}{x}_1\underset{\sim}{y}})}{r_{\underset{\sim}{x}_1\underset{\sim}{y}}}\Big]\Big\{\frac{\exp(-sr_{\underset{\sim}{y}\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}\underset{\sim}{x}_2}} + \gamma_4^{-1}\frac{\partial}{\partial n_{4\underset{\sim}{y}}}\Big[\frac{\exp(-sr_{\underset{\sim}{y}\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}\underset{\sim}{x}_2}}\Big]\Big\}d\underset{\sim}{y} \\
&+\frac{1}{8\pi^2}\int_{S_1^*}\int_{S_1^*} \frac{\exp(-sr_{\underset{\sim}{x}_1\underset{\sim}{y}})}{r_{\underset{\sim}{x}_1\underset{\sim}{y}}} M_{\gamma_1}(\underset{\sim}{y}, \underset{\sim}{y}')\Big\{\frac{\partial}{\partial n_{1\underset{\sim}{y}'}}\Big[\frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}\Big] \\
&+\gamma_1 \frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}\Big\}d\underset{\sim}{y}d\underset{\sim}{y}' \\
&+\frac{1}{8\pi^2}\int_{S_2^*}\int_{S_2^*} \frac{\partial}{\partial n_{2\underset{\sim}{y}}}\Big[\frac{\exp(-sr_{\underset{\sim}{x}_1\underset{\sim}{y}})}{r_{\underset{\sim}{x}_1\underset{\sim}{y}}}\Big] M_{\gamma_2^{-1}}(\underset{\sim}{y}, \underset{\sim}{y}')\Big\{\frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}} \\
&+\gamma_2^{-1}\frac{\partial}{\partial n_{2\underset{\sim}{y}'}}\Big[\frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}\Big]\Big\}d\underset{\sim}{y}d\underset{\sim}{y}' \\
&+\frac{1}{8\pi^2}\int_{S_3^*}\int_{S_3^*} \frac{\exp(-sr_{\underset{\sim}{x}_1\underset{\sim}{y}})}{r_{\underset{\sim}{x}_1\underset{\sim}{y}}} L_{\gamma_3}(\underset{\sim}{y}, \underset{\sim}{y}')\Big\{\frac{\partial}{\partial n_{3\underset{\sim}{y}'}}\Big[\frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}\Big] \\
&+\gamma_3 \frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}\Big\}d\underset{\sim}{y}d\underset{\sim}{y}' \\
&+\frac{1}{8\pi^2}\int_{S_4^*}\int_{S_4^*} \frac{\partial}{\partial n_{4\underset{\sim}{y}}}\Big[\frac{\exp(-sr_{\underset{\sim}{x}_1\underset{\sim}{y}})}{r_{\underset{\sim}{x}_1\underset{\sim}{y}}}\Big] L_{\gamma_4^{-1}}(\underset{\sim}{y}, \underset{\sim}{y}')\Big\{\frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}} \\
&+\gamma_4^{-1}\frac{\partial}{\partial n_{4\underset{\sim}{y}'}}\Big[\frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}\Big]\Big\}d\underset{\sim}{y}d\underset{\sim}{y}' \\
&-\frac{1}{8\pi^2}\int_{S_1^*}\Big\{\int_{S_2^*} \frac{\partial}{\partial n_{2\underset{\sim}{y}}}\Big[\frac{\exp(-sr_{\underset{\sim}{x}_1\underset{\sim}{y}})}{r_{\underset{\sim}{x}_1\underset{\sim}{y}}}\Big] M^*_{\gamma_2^{-1}}(\underset{\sim}{y}, \underset{\sim}{y}')d\underset{\sim}{y}\Big\} \\
&\times\Big\{\frac{\partial}{\partial n_{1\underset{\sim}{y}'}}\Big[\frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}\Big] + \gamma_1 \frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}\Big\}d\underset{\sim}{y}'
\end{aligned}
$$

$$-\frac{1}{8\pi^2}\int_{S_2^*}\Big\{\int_{S_1^*}\frac{\exp(-sr_{\underset{\sim}{x}_1\underset{\sim}{y}})}{r_{\underset{\sim}{x}_1\underset{\sim}{y}}}M^*_{\gamma_1}(\underset{\sim}{y},\underset{\sim}{y}')d\underset{\sim}{y}\Big\}\Big\{\frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}$$

$$+\gamma_2^{-1}\frac{\partial}{\partial n_{2\underset{\sim}{y}'}}[\frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}]\Big\}d\underset{\sim}{y}'$$

$$-\frac{1}{8\pi^2}\int_{S_1^*}\Big\{\int_{S_3^*}\frac{\exp(-sr_{\underset{\sim}{x}_1\underset{\sim}{y}})}{r_{\underset{\sim}{x}_1\underset{\sim}{y}}}L^*_{\gamma_3}(\underset{\sim}{y},\underset{\sim}{y}')d\underset{\sim}{y}\Big\}\Big\{\frac{\partial}{\partial n_{1\underset{\sim}{y}'}}[\frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}]$$

$$+\gamma_1\frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}\Big\}d\underset{\sim}{y}'$$

$$-\frac{1}{8\pi^2}\int_{S_3^*}\Big\{\int_{S_1^*}\frac{\exp(-sr_{\underset{\sim}{x}_1\underset{\sim}{y}})}{r_{\underset{\sim}{x}_1\underset{\sim}{y}}}L_{\gamma_1}(\underset{\sim}{y},\underset{\sim}{y}')d\underset{\sim}{y}\Big\}\Big\{\frac{\partial}{\partial n_{3\underset{\sim}{y}'}}[\frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}]$$

$$+\gamma_3\frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}\Big\}d\underset{\sim}{y}'$$

$$+\frac{1}{8\pi^2}\int_{S_1^*}\Big\{\int_{S_4^*}\frac{\partial}{\partial n_{4\underset{\sim}{y}}}[\frac{\exp(-sr_{\underset{\sim}{x}_1\underset{\sim}{y}})}{r_{\underset{\sim}{x}_1\underset{\sim}{y}}}]M^*_{\gamma_4^{-1}}(\underset{\sim}{y},\underset{\sim}{y}')d\underset{\sim}{y}\Big\}$$

$$\times\Big\{\frac{\partial}{\partial n_{1\underset{\sim}{y}'}}[\frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}]+\gamma_1\frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}\Big\}d\underset{\sim}{y}'$$

$$+\frac{1}{8\pi^2}\int_{S_4^*}\Big\{\int_{S_1^*}\frac{\exp(-sr_{\underset{\sim}{x}_1\underset{\sim}{y}})}{r_{\underset{\sim}{x}_1\underset{\sim}{y}}}L^*_{\gamma_1}(\underset{\sim}{y},\underset{\sim}{y}')d\underset{\sim}{y}\Big\}\Big\{\frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}$$

$$+\gamma_4^{-1}\frac{\partial}{\partial n_{4\underset{\sim}{y}'}}[\frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}]\Big\}d\underset{\sim}{y}'$$

$$+\frac{1}{8\pi^2}\int_{S_2^*}\Big\{\int_{S_3^*}\frac{\exp(-sr_{\underset{\sim}{x}_1\underset{\sim}{y}})}{r_{\underset{\sim}{x}_1\underset{\sim}{y}}}M^*_{\gamma_3}(\underset{\sim}{y},\underset{\sim}{y}')d\underset{\sim}{y}\Big\}\Big\{\frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}$$

$$+\gamma_2^{-1}\frac{\partial}{\partial n_{2\underset{\sim}{y}'}}[\frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}]\Big\}d\underset{\sim}{y}'$$

$$+\frac{1}{8\pi^2}\int_{S_3^*}\Big\{\int_{S_2^*}\frac{\partial}{\partial n_{2\underset{\sim}{y}}}[\frac{\exp(-sr_{\underset{\sim}{x}_1\underset{\sim}{y}})}{r_{\underset{\sim}{x}_1\underset{\sim}{y}}}]L^*_{\gamma_2^{-1}}(\underset{\sim}{y},\underset{\sim}{y}')d\underset{\sim}{y}\Big\}$$

$$\times\Big\{\frac{\partial}{\partial n_{3\underset{\sim}{y}'}}[\frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}]+\gamma_3\frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}\Big\}d\underset{\sim}{y}'$$

$$-\frac{1}{8\pi^2}\int_{S_2^*}\Big\{\int_{S_4^*}\frac{\partial}{\partial n_{4\underset{\sim}{y}}}[\frac{\exp(-sr_{\underset{\sim}{x}_1\underset{\sim}{y}})}{r_{\underset{\sim}{x}_1\underset{\sim}{y}}}]M_{\gamma_4^{-1}}(\underset{\sim}{y},\underset{\sim}{y}')d\underset{\sim}{y}\Big\}$$

$$\times\Big\{\frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}+\gamma_2^{-1}\frac{\partial}{\partial n_{2\underset{\sim}{y}'}}[\frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}]\Big\}d\underset{\sim}{y}'$$

$$-\frac{1}{8\pi^2}\int_{S_4^*}\Big\{\int_{S_2^*}\frac{\partial}{\partial n_{2\underset{\sim}{y}}}[\frac{\exp(-sr_{\underset{\sim}{x}_1\underset{\sim}{y}})}{r_{\underset{\sim}{x}_1\underset{\sim}{y}}}L_{\gamma_2^{-1}}(\underset{\sim}{y},\underset{\sim}{y}')d\underset{\sim}{y}\Big\}$$
$$\times\Big\{\frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}+\gamma_4^{-1}\frac{\partial}{\partial n_{4\underset{\sim}{y}'}}[\frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}]\Big\}d\underset{\sim}{y}'$$
$$-\frac{1}{8\pi^2}\int_{S_3^*}\Big\{\int_{S_4^*}\frac{\partial}{\partial n_{4\underset{\sim}{y}}}[\frac{\exp(-sr_{\underset{\sim}{x}_1\underset{\sim}{y}})}{r_{\underset{\sim}{x}_1\underset{\sim}{y}}}L^*_{\gamma_4^{-1}}(\underset{\sim}{y},\underset{\sim}{y}')d\underset{\sim}{y}\Big\}$$
$$\times\Big\{\frac{\partial}{\partial n_{3\underset{\sim}{y}'}}[\frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}]+\gamma_3\frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}\Big\}d\underset{\sim}{y}'$$
$$-\frac{1}{8\pi^2}\int_{S_4^*}\Big\{\int_{S_3^*}\frac{\exp(-sr_{\underset{\sim}{x}_1\underset{\sim}{y}})}{r_{\underset{\sim}{x}_1\underset{\sim}{y}}}L^+_{\gamma_3}(\underset{\sim}{y},\underset{\sim}{y}')d\underset{\sim}{y}\Big\}\Big\{\frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}$$
$$+\gamma_4^{-1}\frac{\partial}{\partial n_{4\underset{\sim}{y}'}}[\frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}]\Big\}d\underset{\sim}{y}',\tag{4.3}$$

where

$$M_{\gamma_1}(\underset{\sim}{y},\underset{\sim}{y}')=\sum_{\nu=0}^{\infty}K^{(\nu)}_{\gamma_1}(\underset{\sim}{y}',\underset{\sim}{y}),\tag{4.4}$$

$$K^{(0)}_{\gamma_1}(\underset{\sim}{y}',\underset{\sim}{y})=\frac{1}{2\pi}\Big\{\frac{\partial}{\partial n_{1\underset{\sim}{y}}}[\frac{\exp(-sr_{\underset{\sim}{y}\underset{\sim}{y}'})}{r_{\underset{\sim}{y}\underset{\sim}{y}'}}]+\gamma_1\frac{\exp(-sr_{\underset{\sim}{y},\underset{\sim}{y}'})}{r_{\underset{\sim}{y}\underset{\sim}{y}'}}\Big\},\tag{4.5}$$

$$M_{\gamma_2^{-1}}(\underset{\sim}{y},\underset{\sim}{y}')=\sum_{\nu=0}^{\infty}(-1)^\nu K^{(\nu)}_{\gamma_2^{-1}}(\underset{\sim}{y}',\underset{\sim}{y}),\tag{4.6}$$

$$K^{(0)}_{\gamma_2^{-1}}(\underset{\sim}{y}',\underset{\sim}{y})=\frac{1}{2\pi}\Big\{\frac{\partial}{\partial n_{2\underset{\sim}{y}'}}[\frac{\exp(-sr_{\underset{\sim}{y}\underset{\sim}{y}'})}{r_{\underset{\sim}{y}\underset{\sim}{y}'}}+\gamma_2^{-1}\frac{\partial^2}{\partial n_{2\underset{\sim}{y}}\partial n_{2\underset{\sim}{y}'}}[\frac{\exp(-sr_{\underset{\sim}{y}\underset{\sim}{y}'})}{r_{\underset{\sim}{y}\underset{\sim}{y}'}}]\Big\},\tag{4.7}$$

$$L_{\gamma_3}(\underset{\sim}{y},\underset{\sim}{y}')=\sum_{\nu=0}^{\infty}(-1)^\nu K^{(\nu)}_{\gamma_3}(\underset{\sim}{y}',\underset{\sim}{y}),\tag{4.8}$$

where $K^{(0)}_{\gamma_3}(\underset{\sim}{y}',\underset{\sim}{y})$ has the same form (4.5) with the interchanges $\gamma_1\leftrightarrow\gamma_3$ and $\underset{\sim}{n}_1\leftrightarrow\underset{\sim}{n}_3$,

$$L_{\gamma_4^{-1}}(\underset{\sim}{y},\underset{\sim}{y}')=\sum_{\nu=0}^{\infty}K^{(\nu)}_{\gamma_4^{-1}}(\underset{\sim}{y}',\underset{\sim}{y}),\tag{4.9}$$

where $K^{(0)}_{\gamma_4^{-1}}(\underset{\sim}{y}',\underset{\sim}{y})$ has the same form (4.7) with the interchanges $\gamma_2\leftrightarrow\gamma_4$ and $\underset{\sim}{n}_2\leftrightarrow\underset{\sim}{n}_4$,

$$M^*_{\gamma_2^{-1}}(\underset{\sim}{y},\underset{\sim}{y}')=\sum_{\nu=0}^{\infty}{}^*K^{(\nu)}_{\gamma_2^{-1}}(\underset{\sim}{y}',\underset{\sim}{y}),\tag{4.10}$$

$$^*K^{(0)}_{\gamma_2^{-1}}(\underset{\sim}{y}',\underset{\sim}{y}) = \frac{1}{2\pi}\Big\{\frac{\exp(-sr_{\underset{\sim}{y}\underset{\sim}{y}'})}{r_{\underset{\sim}{y}\underset{\sim}{y}'}} + \gamma_2^{-1}\frac{\partial}{\partial n_{2\underset{\sim}{y}}}[\frac{\exp(-sr_{\underset{\sim}{y}\underset{\sim}{y}'})}{r_{\underset{\sim}{y}\underset{\sim}{y}'}}]\Big\}, \tag{4.11}$$

$$M^*_{\gamma_1}(\underset{\sim}{y},\underset{\sim}{y}') = \sum_{\nu=0}^{\infty}(-1)^{\nu}{}^*K^{(\nu)}_{\gamma_1}(\underset{\sim}{y}',\underset{\sim}{y}), \tag{4.12}$$

$$^*K^{(0)}_{\gamma_1}(\underset{\sim}{y}',\underset{\sim}{y}) = \frac{1}{2\pi}\Big\{\frac{\partial^2}{\partial n_{1\underset{\sim}{y}}\partial n_{2\underset{\sim}{y}'}}[\frac{\exp(-sr_{\underset{\sim}{y}\underset{\sim}{y}'})}{r_{\underset{\sim}{y}\underset{\sim}{y}'}}] + \gamma_1\frac{\partial}{\partial n_{1\underset{\sim}{y}}}[\frac{\exp(-sr_{\underset{\sim}{y}\underset{\sim}{y}'})}{r_{\underset{\sim}{y}\underset{\sim}{y}'}}]\Big\}, \tag{4.13}$$

$$L^*_{\gamma_3}(\underset{\sim}{y},\underset{\sim}{y}') = \sum_{\nu=0}^{\infty}{}^*K^{(\nu)}_{\gamma_3}(\underset{\sim}{y}',\underset{\sim}{y}), \tag{4.14}$$

$$^*K^{(0)}_{\gamma_3}(\underset{\sim}{y}',\underset{\sim}{y}) = \frac{1}{2\pi}\Big\{\frac{\partial^2}{\partial n_{3\underset{\sim}{y}}\partial n_{1\underset{\sim}{y}'}}[\frac{\exp(-sr_{\underset{\sim}{y}\underset{\sim}{y}'})}{r_{\underset{\sim}{y}\underset{\sim}{y}'}}] + \gamma_3\frac{\partial}{\partial n_{3\underset{\sim}{y}}}[\frac{\exp(-sr_{\underset{\sim}{y}\underset{\sim}{y}'})}{r_{\underset{\sim}{y}\underset{\sim}{y}'}}]\Big\}, \tag{4.15}$$

$$L_{\gamma_1}(\underset{\sim}{y},\underset{\sim}{y}') = \sum_{\nu=0}^{\infty}(-1)^{\nu}K^{(\nu)}_{\gamma_1}(\underset{\sim}{y}',\underset{\sim}{y}), \tag{4.16}$$

where $K^{(0)}_{\gamma_1}(\underset{\sim}{y}',\underset{\sim}{y})$ has the same form (4.5),

$$M^*_{\gamma_4^{-1}}(\underset{\sim}{y},\underset{\sim}{y}') = \sum_{\nu=0}^{\infty}{}^*K^{(\nu)}_{\gamma_4^{-1}}(\underset{\sim}{y}',\underset{\sim}{y}), \tag{4.17}$$

where $^*K^{(0)}_{\gamma_4^{-1}}(\underset{\sim}{y}',\underset{\sim}{y})$ has the same form (4.11) with the interchanges $\gamma_2 \leftrightarrow \gamma_4$ and $\underset{\sim}{n}_2 \leftrightarrow \underset{\sim}{n}_4$,

$$L^*_{\gamma_1}(\underset{\sim}{y},\underset{\sim}{y}') = \sum_{\nu=0}^{\infty}{}^+K^{(\nu)}_{\gamma_1}(\underset{\sim}{y}',\underset{\sim}{y}), \tag{4.18}$$

$$^+K^{(0)}_{\gamma_1}(\underset{\sim}{y}',\underset{\sim}{y}) = \frac{1}{2\pi}\Big\{\frac{\partial^2}{\partial n_{1\underset{\sim}{y}}\partial n_{4\underset{\sim}{y}'}}[\frac{\exp(-sr_{\underset{\sim}{y}\underset{\sim}{y}'})}{r_{\underset{\sim}{y}\underset{\sim}{y}'}}] + \gamma_1\frac{\partial}{\partial n_{1\underset{\sim}{y}}}[\frac{\exp(-sr_{\underset{\sim}{y}\underset{\sim}{y}'})}{r_{\underset{\sim}{y}\underset{\sim}{y}'}}]\Big\}, \tag{4.19}$$

$$M^*_{\gamma_3}(\underset{\sim}{y},\underset{\sim}{y}') = \sum_{\nu=0}^{\infty}(-1)^{\nu}{}^{**}K^{(\nu)}_{\gamma_3}(\underset{\sim}{y}',\underset{\sim}{y}), \tag{4.20}$$

where ${}^{**}K^{(0)}_{\gamma_3}(\underset{\sim}{y}', \underset{\sim}{y})$ has the same form (4.15) with the interchanges $\underset{\sim}{n}_1 \leftrightarrow \underset{\sim}{n}_2$,

$$L^{*}_{\gamma_2^{-1}}(\underset{\sim}{y}, \underset{\sim}{y}') = \sum_{\nu=0}^{\infty} (-1)^{\nu} {}^{*}K^{(\nu)}_{\gamma_2^{-1}}(\underset{\sim}{y}', \underset{\sim}{y}), \tag{4.21}$$

where ${}^{*}K^{(0)}_{\gamma_2^{-1}}(\underset{\sim}{y}', \underset{\sim}{y})$ has the same form (4.11)

$$M_{\gamma_4^{-1}}(\underset{\sim}{y}, \underset{\sim}{y}') = \sum_{\nu=0}^{\infty} (-1)^{\nu} K^{(\nu)}_{\gamma_4^{-1}}(\underset{\sim}{y}', \underset{\sim}{y}), \tag{4.22}$$

where $K^{(0)}_{\gamma_4^{-1}}(\underset{\sim}{y}', \underset{\sim}{y})$ has the same form (4.7) with the interchanges $\gamma_2 \leftrightarrow \gamma_4$ and $\underset{\sim}{n}_2 \leftrightarrow \underset{\sim}{n}_4$,

$$L_{\gamma_2^{-1}}(\underset{\sim}{y}, \underset{\sim}{y}') = \sum_{\nu=0}^{\infty} K^{(\nu)}_{\gamma_2^{-1}}(\underset{\sim}{y}', \underset{\sim}{y}), \tag{4.23}$$

where $K^{(0)}_{\gamma_2^{-1}}(\underset{\sim}{y}', \underset{\sim}{y})$ has the same form (4.7)

$$L^{*}_{\gamma_4^{-1}}(\underset{\sim}{y}, \underset{\sim}{y}') = \sum_{\nu=0}^{\infty} (-1)^{\nu} {}^{*}K^{(\nu)}_{\gamma_4^{-1}}(\underset{\sim}{y}', \underset{\sim}{y}), \tag{4.24}$$

where ${}^{*}K^{(0)}_{\gamma_4^{-1}}(\underset{\sim}{y}', \underset{\sim}{y})$ has the same form (4.11) with the interchanges $\gamma_2 \leftrightarrow \gamma_4$ and $\underset{\sim}{n}_2 \leftrightarrow \underset{\sim}{n}_4$,

$$L^{+}_{\gamma_3}(\underset{\sim}{y}, \underset{\sim}{y}') = \sum_{\nu=0}^{\infty} {}^{+}K^{(\nu)}_{\gamma_3}(\underset{\sim}{y}', \underset{\sim}{y}), \tag{4.25}$$

where ${}^{+}K^{(0)}_{\gamma_3}(\underset{\sim}{y}', \underset{\sim}{y})$ has the same form (4.19) with the interchanges $\gamma_1 \leftrightarrow \gamma_3$ and $\underset{\sim}{n}_1 \leftrightarrow \underset{\sim}{n}_3$. In these formulae, we note that $K^{(\nu)}_{\gamma_i}(\underset{\sim}{y}', \underset{\sim}{y})$ being the iterates of the kernel $K^{(0)}_{\gamma_i}(\underset{\sim}{y}', \underset{\sim}{y})$, $(i = 1, 2, 3, 4)$.
Similarly, we can find $\overline{\chi}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2)$ for the other fifteen cases.

On the basis of (4.3), the function $\overline{\chi}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2)$ will be estimated for $s \longrightarrow \infty$ together with small/large impedances γ_j $(j = 1, 2, 3, 4)$. The case when $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ lie in the neighbourhood of the parts S_1^*, S_2^* of the inner bounding surface $\tilde{S}_1$ or in the neighbourhood of the parts S_3^*, S_4^* of the outer bounding surface $\tilde{S}_2$ is particularly interesting. In what follows, we shall use coordinates similar to those obtained in Zayed [6, 9, 10] to examine this case.

5. Differential Geometry of the Boundaries

Let $h_j > 0$ $(j = 1, 2, 3, 4)$ be sufficiently small. Let $n_j (j = 1, 2, 3, 4)$ be the minimum distances from a point $\underset{\sim}{x} = (x^1, x^2, x^3)$ of the domain Ω to the parts S_j^* $(j = 1, 2, 3, 4)$

respectively. Let $\underset{\sim}{n}_j(\sigma_1)$, $(j=1,2)$ denote the inward drawn unit normals to the boundries S_j^*, $(j=1,2)$ of the inner bounding surface $\tilde{S}_1$ and let $\underset{\sim}{n}_j(\sigma_2)(j=3,4)$ denote the inward drawn unit normals to the parts S_j^*, $(j=3,4)$ of the outer bounding surface $\tilde{S}_2$ respectively. Then, we note that the coordinates in the neighbourhood of the parts $S_j^*(j=1,2)$ of $\tilde{S}_1$ are in the same form as in Section 5.2 of Zayed [9] with the interchanges $n_1 \leftrightarrow n_j$, $h_1 \leftrightarrow h_j$, $I_1 \leftrightarrow I_j$, $\mathcal{D}(I_1) \leftrightarrow \mathcal{D}(I_j)$ and $\delta_1 \leftrightarrow \delta_j$ $(j=1,2)$. Thus, we have the same formulae (5.2.1) - (5.1.5) of Section 5.2. in Zayed [9] with the interchanges $n_1 \leftrightarrow n_j$, $\underset{\sim}{n}_1(\sigma_1) \leftrightarrow \underset{\sim}{n}_j(\sigma_1)$, $(j=1,2)$. Similarly, the coordinates in the neighbourhood of the parts S_j^* $(j=3,4)$ of S_2^* are similar to those obtained in Section 5.1. of Zayed [9] with the interchanges $n_2 \leftrightarrow n_j$, $h_2 \leftrightarrow h_j$, $I_2 \leftrightarrow I_i$, $\mathcal{D}(I_2) \leftrightarrow \mathcal{D}(I_j)$ and $\delta_2 \leftrightarrow \delta_j$ $(j=3,4)$. Thus, we have the same formulae (5.1.1) - (5.1.6) of Section 5.1. in Zayed [9] with the interchanges $n_2 \leftrightarrow n_j$, $\underset{\sim}{n}_2(\sigma_2) \leftrightarrow \underset{\sim}{n}_j(\sigma_2)$, $(j=3,4)$.

6. Some Local Expansions

It now follows that the local expansions of the functions:

$$\frac{\exp(-sr_{\underset{\sim}{x}\underset{\sim}{y}})}{r_{\underset{\sim}{x}\underset{\sim}{y}}},\ \frac{\partial}{\partial n_{j\underset{\sim}{y}}}\left[\frac{\exp(-sr_{\underset{\sim}{x}\underset{\sim}{y}})}{r_{\underset{\sim}{x}\underset{\sim}{y}}}\right],\ (j=1,2,3,4), \tag{6.1}$$

when the distance between $\underset{\sim}{x}$ and $\underset{\sim}{y}$ is small, are very similar to those obtained in Sections 4 and 5 of Zayed [6]. Consequently, for small/large impednaces γ_j $(j=1,2,3,4)$ the local expansions the following kernels:

$$K_{\gamma_j}^{(0)}(\underset{\sim}{y}',\underset{\sim}{y}),\ {}^*K_{\gamma_j}^{(0)}(\underset{\sim}{y}',\underset{\sim}{y}),\ {}^+K_{\gamma_j}^{(0)}(\underset{\sim}{y}',\underset{\sim}{y}),\ (j=1,3),$$
$$ {}^{**}K_{\gamma_3}(\underset{\sim}{y}',\underset{\sim}{y}), \tag{6.2}$$
$$K_{\gamma_j^{-1}}^{(0)}(\underset{\sim}{y}',\underset{\sim}{y}),\ {}^*K_{\gamma_j^{-1}}^{(0)}(\underset{\sim}{y}',\underset{\sim}{y})\quad (j=2,4) \tag{6.3}$$

when the distance between $\underset{\sim}{y}$ and $\underset{\sim}{y}'$ is small, follows directly from the knowledge of the local expansions of the functions (6.1).

Definition 1. Let $\underset{\sim}{\xi}_1$ and $\underset{\sim}{\xi}_2$ be points in the half-part $\xi^3>0$, of the (ξ^1,ξ^2,ξ^3)-space. Define

$$\rho_{12}=\sqrt{(\xi_1^1-\xi_2^1)^2+(\xi_1^2-\xi_2^2)^2+(\xi_1^3+\xi_2^3)^2}$$

An $e^\lambda(\underset{\sim}{\xi}_1,\underset{\sim}{\xi}_2;s)$-function is defined for points $\underset{\sim}{\xi}_1$ and $\underset{\sim}{\xi}_2$ belong to sufficiently small domains $\mathcal{D}(I_j)$ $(j=1,2,3,4)$ except when $\underset{\sim}{\xi}_1=\underset{\sim}{\xi}_2\in I_j$ $(j=1,2,3,4)$, where λ is called the degree of this function. For every positive integer $\wedge$, it has the local expansion (see [6], [9]):

$$e^\lambda(\underset{\sim}{\xi}_1,\underset{\sim}{\xi}_2;s)=\Sigma^* f(\xi_1^1,\xi_1^2)(\xi_1^3)^{P_1}(\xi_2^3)^{P_2}\left(\frac{\partial}{\partial\xi_1^1}\right)^{l_1}\left(\frac{\partial}{\partial\xi_1^2}\right)^{l_2}\left(\frac{\partial}{\partial\xi_1^3}\right)^{l_3}$$
$$\times\frac{\exp(-s\rho_{12})}{\rho_1 2}+R^\wedge(\underset{\sim}{\xi}_1,\underset{\sim}{\xi}_2;s), \tag{6.4}$$

where Σ^* denotes a sum of a finite number of terms in which $f(\xi_1^1, \xi_1^2)$ is an infinitely differentiable functions. In this expansion P_1, P_2, l_1, l_2, l_3 are integers, where $P_1 \geq 0$, $P_2 \geq 0$, $l_1 \geq 0$, $l_2 \geq 0$, $\lambda = \min(P_1 + P_2 - q)$, $q = l_1 + l_2 + l_3$ and the minimum is taken over all terms which occur in the summation Σ^*. The remainder $R^\wedge(\underset{\sim}{\xi}_1, \underset{\sim}{\xi}_2; s)$ has continuous derivatives of order $d \leq \wedge$ satisfying

$$D^d R^\wedge(\underset{\sim}{\xi}_1, \underset{\sim}{\xi}_2; s) = O(s^{-\wedge} \exp(-A s \rho_{12}) \quad \text{as } s \to \infty, \tag{6.5}$$

where A is a positive constant.

Thus, using methods similar to those obtained in Sections 6-10 of Zayed [6], we can show that the functions (6.1) are e^λ-functions with degrees $\lambda = -1, -2$, respectively. Consequently, for small impedances γ_j $(j = 1, 3)$ the functions (6.2) are e^λ-functions with degrees $\lambda = 0, -1, -1, -1$, respectively, while for large impedances λ_j $(j = 2, 4)$ the functions (6.3) are e^λ-functions with degrees $\lambda = 0, 1$, respectively (see also [8]).

Definition 2. If $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ are points in large domains $\Omega + S_j^*(j = 1, 2, 3, 4)$, then we define

$$r_{12} = \min_{\underset{\sim}{y}}(r_{\underset{\sim}{x}_1 \underset{\sim}{y}} + r_{\underset{\sim}{x}_2 \underset{\sim}{y}}) \quad \text{if } \underset{\sim}{y} \in S_1^*,$$

$$R_{12} = \min_{\underset{\sim}{y}}(r_{\underset{\sim}{x}_1 \underset{\sim}{y}} + r_{\underset{\sim}{x}_2 \underset{\sim}{y}}) \quad \text{if } \underset{\sim}{y} \in S_2^*,$$

$$r_{12}^* = \min_{\underset{\sim}{y}}(r_{\underset{\sim}{x}_1 \underset{\sim}{y}} + r_{\underset{\sim}{x}_2 \underset{\sim}{y}}) \quad \text{if } \underset{\sim}{y} \in S_3^*,$$

and

$$R_{12}^* = \min_{\underset{\sim}{y}}(r_{\underset{\sim}{x}_1 \underset{\sim}{y}} + r_{\underset{\sim}{x}_2 \underset{\sim}{y}}) \quad \text{if } \underset{\sim}{y} \in S_4^*,$$

An $E^\lambda(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2)$-function is defined and infinitely differentiable with respect to $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ when these points belong to large domains $\Omega + S_j^*(j = 1, 2, 3, 4)$ except when $\underset{\sim}{x}_1 = \underset{\sim}{x}_2 \in S_j^*$ $(j = 1, 2, 3, 4)$. Thus, the E^λ-function has a similar local expansion of the e^λ-function (see [6], [9], [10]).

With the help of Section 8 and 9 in Zayed [6], it is easily seen that the formula (4.3) is an $E^{-2}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s)$-function and consequently

$$\begin{aligned} \overline{G}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2) = {} & O\Big\{\frac{\exp(-A_1 s r_{12})}{r_{12}^2}\Big\} + O\Big\{\frac{\exp(-A_2 s R_{12})}{R_{12}^2}\Big\} \\ & + O\Big\{\frac{\exp(-A_3 s r_{12})}{r_{12}^{*2}}\Big\} + O\Big\{\frac{\exp(-A_4 s R_{12})}{R_{12}^{*2}}\Big\}, \end{aligned} \tag{6.6}$$

which is valid for $s \to \infty$ and for small/large impedances γ_j $(j = 1, 2, 3, 4)$ as indicated in the specification of case 1, where A_j $(j = 1, 2, 3, 4)$ are positive constants. Formula (6.6) shows that $\overline{G}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2)$ is exponentially small for $s \to \infty$. Similar statements are true in the other fifteen cases.

With reference to Section 10 in Zayed [6], if the e^λ-expansions of the functions (6.1)-(6.3) are introduced into (4.3) and if we use formulae similar to (6.4) and (6.9) of Section

6 in Zayed [6], we obtain the following local behaviour of $\overline{\chi}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2)$ when r_{12}, R_{12}, r^*_{12} and R^*_{12} are small, which is valid for $s \to \infty$ and for small γ_1, γ_3 and large γ_2, γ_4:

$$\overline{\chi}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2) = \sum_{j=1}^{4} \overline{\chi}_j(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2), \tag{6.7}$$

where

(a) if $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ belong to a sufficiently small domain $\mathcal{D}(I_1)$, then

$$\overline{\chi}_1(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2) = -\frac{1}{8\pi}\Big\{1 - \gamma_1\big(\frac{\partial}{\partial \xi^3_1}\big)^{-1}\Big\}\frac{\exp(-s\rho_{12})}{\rho_{12}} + O\Big\{\frac{\exp(-A_1 s\rho_{12})}{\rho_{12}}\Big\}, \tag{6.8}$$

(b) if $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ belong to a sufficiently small domain $\mathcal{D}(I_2)$, then

$$\overline{\chi}_2(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2) = \frac{1}{8\pi}\Big\{1 - \gamma_2^{-1}\big(\frac{\partial}{\partial \xi^3_1}\big)\Big\}\frac{\exp(-s\rho_{12})}{\rho_{12}} + O\Big\{\frac{\exp(-A_2 s\rho_{12})}{\rho_{12}}\Big\}, \tag{6.9}$$

(c) if $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ belong to a sufficiently small domain $\mathcal{D}(I_3)$, then

$$\overline{\chi}_3(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2) = \frac{1}{8\pi}\Big\{1 - \gamma_3\big(\frac{\partial}{\partial \xi^3_1}\big)^{-1}\Big\}\frac{\exp(-s\rho_{12})}{\rho_{12}} + O\Big\{\frac{\exp(-A_3 s\rho_{12})}{\rho_{12}}\Big\}, \tag{6.10}$$

(d) if $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ belong to a sufficiently small domain $\mathcal{D}(I_4)$, then

$$\overline{\chi}_4(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2) = -\frac{1}{8\pi}\Big\{1 - \gamma_4^{-1}\big(\frac{\partial}{\partial \xi^3_1}\big)\Big\}\frac{\exp(-s\rho_{12})}{\rho_{12}} + O\Big\{\frac{\exp(-A_4 s\rho_{12})}{\rho_{12}}\Big\}, \tag{6.11}$$

When $r_{12} \geq \delta_1 > 0$, $R_{12} \geq \delta_2 > 0$, $r^*_{12} \geq \delta_3 > 0$ and $R^*_{12} \geq \delta_4 > 0$ the function $\overline{\chi}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2)$ is of order $O\{\exp(-Bs)\}$ as $s \to \infty$, where B is a positive constant. Thus, since $\lim\limits_{r_{12}\to 0} \frac{r_{12}}{\rho_{12}} = \lim\limits_{R_{12}\to 0} \frac{R_{12}}{\rho_{12}} = \lim\limits_{r^*_{12}\to 0} \frac{r^*_{12}}{\rho_{12}} = \lim\limits_{R^*_{12}\to 0} \frac{R^*_{12}}{\rho_{12}} = 1$ (see [6], [9]), then we have the asymptotic formulae (6.8)-(6.11) with ρ_{12} in the small domains $\mathcal{D}(I_j)$ $(j = 1, 2, 3, 4)$ being replaced by r_{12}, R_{12}, r^*_{12} and R^*_{12} in the large domains $\Omega + S^*_j$ $(j = 1, 2, 3, 4)$ repsectively.

Similar formulae for the other fifteen cases can be found.

7. Construction of Our Results

Since, for $\xi^3 \geq h_j > 0$ $(j = 1, 2, 3, 4)$, the functions $\overline{\chi}_j(\underset{\sim}{x}, \underset{\sim}{x}; s^2)$ are of order $O\Big\{\exp(-2sA_j h_j)\Big\}$, $(j = 1, 2, 3, 4)$, the integral over the region Ω of the function $\overline{\chi}(\underset{\sim}{x}, \underset{\sim}{x}; s^2)$ can be approximated in the following way (see (3.12)):

$$\overline{R}(s^2) = \sum_{j=3}^{4} \int_{S^*_j} \int_{\xi^3=0}^{h_j} \overline{\chi}_j(\underset{\sim}{x}, \underset{\sim}{x}; s^2)\Big\{1 - 2\xi^3 H_1 + (\xi^3)^2 N_1\Big\} d\xi^3 dS^*_j$$

$$-\sum_{j=1}^{2}\int_{S_j^*}\int_{\xi^3=0}^{h_j}\overline{\chi}_j(\underset{\sim}{x},\underset{\sim}{x};s^2)\Big\{1+2\xi^3H_1^*+(\xi^3)^2N_1^*\Big\}d\xi^3dS_j^*$$

$$+\sum_{j=1}^{4}O\Big\{\exp(-2sA_jh_j)\Big\} \quad \text{as} \quad s\to\infty. \tag{7.1}$$

If the e^{λ}-expansions of $\overline{\chi}_j(\underset{\sim}{x},\underset{\sim}{x};s^2)$ $(j=1,2,3,4)$ are introduced into (7.1), and with the help of the formula (10.2) of Section 10 in Zayed [9], (see, also [6]) we deduce, after inverting Laplace transforms and using (3.7), that our results (2.5)-(2.10) have been constructed.

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