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HADAMARD PRODUCTS OF CERTAIN MEROMORPHIC UNIVALENT FUNCTIONS

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Abstract. The object of the present paper is to show convolution properties, order of starlikeness, integral transforms and the extreme points for certain classes of meromorphic univalent functions having postive coefficients. All of the results are sharp.

1. Introduction

Let Σ denote the class of functions of the form

$$f(z)=\frac{1}{z}+\sum_{k=1}^{\infty}a_kz^k \tag{1.1}$$

which are regular and univalent in the punctured disk $E=\{z:0<|z|<1\}$. A function f belonging to Σ is said to be meromorphically starlike of order α if it satisfies

$$-Re\left\{\frac{zf(z)}{f(z)}\right\}>\alpha$$

for some $\alpha(0\le\alpha<1)$ and all $z\in U=\{z:|z|<1\}$. We denote by $\Sigma^*(\alpha)$ the class of all meromorphically starlike functions of order α.

The class $\Sigma^*(\alpha)$ and related classes have been extensively studied by Bajpai[2], Clunie [4], Pommerenke [7], Morgra, Reddy and Juneja [6] and others ([1], [3]).

The Hadamard product or convolution of two functions f,g in Σ will be denoted by $f*g$. Robertson [8] has shown that if $f,g\in\Sigma$, then so is their convolution $f*g$.

Let $\Sigma(\lambda,\alpha,\beta,\gamma)$ denote the class of functions f in Σ satisfying the condition

$$Re\left|z^2(D^\lambda f(z))'+1\right|<\beta\left|(2\gamma-1)z^2(D^\lambda f(z))'+(2\alpha\gamma-1)\right|$$

for some $\alpha(0\le\alpha<1)$, $\beta(0<\beta\le1)$, $\gamma(\frac{1}{2}\le\gamma\le1)$ and for all $z\in U$, where $D^\lambda:\Sigma\to\Sigma$ is the operator defined by

$$D^\lambda f(z)=\frac{1}{z(1-z)^{\lambda+1}}*f(z)(\lambda>-1). \tag{1.2}$$

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Notice that from the identity

$$\frac{1}{z(1-z)^{\lambda+2}} = \frac{1}{z(1-z)^{\lambda+1}} * \left(\frac{\lambda+2}{(\lambda+1)z(1-z)} + \frac{2z-1}{(\lambda+1)z(1-z)^2}\right),$$

we get

$$z(D^\lambda f(z))' = (\lambda+1)D^{\lambda+1}f(z) - (\lambda+2)D^\lambda f(z)(\lambda > -1). \tag{1.3}$$

For $\lambda = n \in N_0 = \{0, 1, 2, \ldots\}$, we note [9] that the relation (1.2) may be expressed as

$$D^n f(z) = \frac{1}{z}\left(\frac{z^{n+1}f(z)}{n!}\right)^{(n)}. \tag{1.4}$$

Let Σ_p, $\Sigma_p^*(\alpha)$ and $\Sigma_p(\lambda, \alpha, \beta, \gamma)$ denote the subclasses of Σ, $\Sigma^*(\alpha)$ and $\Sigma(\lambda, \alpha, \beta, \gamma)$, respectively, whose elements can be expressed in the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k \quad (a_k \geq 0, z \in U). \tag{1.5}$$

In particular, the classes $\Sigma_p(0, \alpha, 1, 1)$ and $\Sigma_p(0, \alpha, \beta, \gamma)$ are introduced by Aouf[1] and Cho, Lee and Owa[3], respectively.

For the classes $\Sigma_p(0, \alpha, \beta, \gamma)$ and $\Sigma_p^*(\alpha)$, Cho, Lee and Owa[3] and Mogra, Reddy and Juneja [6] proved the following results.

Theorem A. *Let f be of the form* (1.5) *and $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $\frac{1}{2} \leq \gamma \leq 1$. Then $f \in \Sigma_p(0, \alpha, \beta, \gamma)$ if and only if*

$$\sum_{k=1}^{\infty} k(1 + 2\beta\gamma - \beta)a_k \leq 2\beta\gamma(1-\alpha).$$

Theorem B. *The extreme points of $\Sigma_p(0, \alpha, \beta, \gamma)(0 \leq \alpha < 1,\ 0 < \beta \leq 1,\ \frac{1}{2} \leq \gamma \leq 1)$ are the functions given by $f_0(z) = \frac{1}{z}$ and*

$$f_k(z) = \frac{1}{z} + \frac{2\beta\gamma(1-\alpha)}{k(1+2\beta\gamma-\beta)}z^k \quad (k = 1, 2, \ldots).$$

Theorem C. *Let f be of the form (1.5) and $0 \leq \alpha < 1$. Then $f \in \Sigma_p^*(\alpha)$ if and only if*

$$\sum_{k=1}^{\infty}(k+\alpha)a_k \leq 1 - \alpha.$$

In this paper, we obtain certain properties of $f * g$ when $f \in \Sigma_p(\lambda, \alpha, \beta, \gamma)$ and $g \in \Sigma_p(\mu, \delta, \beta, \gamma)$ for $\lambda, \mu \geq 0$ and $0 \leq \alpha < 1$, $0 \leq \delta < 1$, $0 < \beta \leq 1$, $\frac{1}{2} \leq \gamma \leq 1$. We also determine the order of starlikeness and consider integral transforms for the functions in $\Sigma_p(\lambda,\alpha,\beta,\gamma)$. Further, we find extreme points of the class $\Sigma_p(\lambda,\alpha,\beta,\gamma)$.

2. Convolution Properties

We first prove the following result which will be used heavily in the paper.

Lemma 1. *Let f of the form (1.5) be regular in U, $\lambda > -1$, $0 \le \alpha < 1$, $0 < \beta \le 1$ and $\frac{1}{2} \le \gamma \le 1$. Then $f \in \Sigma_p(\lambda,\alpha,\beta,\gamma)$ if and only if*

$$\sum_{k=1}^{\infty} k(1+2\beta\gamma-\beta)B_k(\lambda)a_k \le 2\beta\gamma(1-\alpha) \tag{2.1}$$

where

$$B_k(\lambda) = \frac{(\lambda+1)(\lambda+2)\cdots(\lambda+k+1)}{(k+1)!} \tag{2.2}$$

Proof. Since

$$D^\lambda f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} B_k(\lambda)a_k z^k, \tag{2.3}$$

in view of (1.3) and the definition of $\Sigma_p(\lambda,\alpha,\beta,\gamma)$, we have

$$f \in \Sigma_p(\lambda,\alpha,\beta,\gamma) \Longleftrightarrow D^\lambda f \in \Sigma_p(0,\alpha,\beta,\gamma), \tag{2.4}$$

and so Lemma 1 follows immediately from Theorem A.

Corollary 1. *Let f be of the form (1.5), $\lambda > -1$, $0 \le \alpha < 1$, $0 < \beta \le 1$ and $\frac{1}{2} \le \gamma \le 1$. If $f \in \Sigma_p(\lambda,\alpha,\beta,\gamma)$, then*

$$a_k \le \frac{2\beta\gamma(1-\alpha)}{k(1+2\beta\gamma-\beta)B_k(\lambda)} (k \ge 1),$$

with equality for the functions of the form

$$F_k(z) = \frac{1}{z} + \frac{2\beta\gamma(1-\alpha)}{k(1+2\beta\gamma-\beta)B_k(\lambda)} z^k.$$

For the next theorem and its corollaries, we assume that f is given by (1.5) and g by

$$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k \ (b_k \ge 0, z \in U). \tag{2.5}$$

Theorem 1. *Let $\lambda \ge 0$, $\mu \ge 0$, $\nu = 0$ or 1 and $0 \le \alpha < 1$, $0 \le \delta < 1$. If $f \in \Sigma_p(\lambda,\alpha,\beta,\gamma)$ and $g \in \Sigma_p(\mu,\delta,\beta,\gamma)$, then $f * g \in \Sigma_p(\nu,\rho,\beta,\gamma)$, where*

$$\rho = \frac{(1+2\beta\gamma-\beta)(\lambda+1)(\lambda+2)(\mu+1)(\mu+2) - 4\beta\gamma(1-\alpha)(1-\delta)(\nu+1)(\nu+2)}{(1+2\beta\gamma-\beta)(\lambda+1)(\lambda+2)(\mu+1)(\mu+2)} \tag{2.6}$$

Proof. We need to find the largest $\rho = \rho(\alpha, \delta, \lambda, \mu, \nu, \beta, \gamma)$ for which

$$\sum_{k=1}^{\infty} B_k(\lambda) \frac{k(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)} a_k \le 1 \tag{2.7}$$

and

$$\sum_{k=1}^{\infty} B_k(\mu) \frac{k(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\delta)} b_k \le 1 \tag{2.8}$$

imply that

$$\sum_{k=1}^{\infty} B_k(\nu) \frac{k(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\rho)} a_k b_k \le 1. \tag{2.9}$$

Using the Cauchy-Schwarz inequality, (2.7) and (2.8) together yield

$$\sum_{k=1}^{\infty} \frac{k(1+2\beta\gamma-\beta)}{2\beta\gamma} \left(\frac{B_k(\lambda)B_k(\mu)}{(1-\alpha)(1-\delta)} \right)^{\frac{1}{2}} \sqrt{a_k b_k} \le 1. \tag{2.10}$$

Thus it is sufficient to show that

$$\frac{B_k(\nu)}{1-\rho} a_k b_k \le \left(\frac{B_k(\lambda)B_k(\mu)}{(1-\alpha)(1-\delta)} \right)^{\frac{1}{2}} \sqrt{a_k b_k}$$

for all $\rho \le \rho(\alpha, \delta, \lambda, \mu, \nu, \beta, \gamma)$ and each $k \ge 1$. Equivalently, we need to prove that

$$\sqrt{a_k b_k} \le \left(\frac{B_k(\lambda)B_k(\mu)}{(1-\alpha)(1-\delta)} \right)^{\frac{1}{2}} \frac{1-\rho}{B_k(\nu)} \tag{2.11}$$

for all $\rho \le \rho(\alpha, \delta, \lambda, \mu, \nu, \beta, \gamma)$ and each $k \ge 1$. On the other hand, it follows from (2.10) that

$$\sqrt{a_k b_k} \le \frac{2\beta\gamma}{k(1+2\beta\gamma-\beta)} \left(\frac{(1-\alpha)(1-\delta)}{B_k(\lambda)B_k(\mu)} \right)^{\frac{1}{2}} \tag{2.12}$$

for each $k \ge 1$. Therefore, in view of (2.11) and (2.12), it is sufficient to establish that

$$\frac{2\beta\gamma}{k(1+2\beta\gamma-\beta)} \left(\frac{(1-\alpha)(1-\delta)}{B_k(\lambda)B_k(\mu)} \right)^{\frac{1}{2}} \le \left(\frac{B_k(\lambda)B_k(\mu)}{(1-\alpha)(1-\delta)} \right)^{\frac{1}{2}} \frac{1-\rho}{B_k(\nu)},$$

that is, that

$$\frac{2\beta\gamma(1-\alpha)(1-\delta)B_k(\nu)}{k(1+2\beta\gamma-\beta)B_k(\lambda)B_k(\mu)} \le 1-\rho,$$

which is equivalent to

$$\rho \le 1 - \frac{2\beta\gamma(1-\alpha)(1-\delta)B_k(\nu)}{k(1+2\beta\gamma-\beta)B_k(\lambda)B_k(\mu)}. \tag{2.13}$$

Note that

$$B_k(\nu) = \begin{cases} 1 & \text{if } \nu = 0, \\ k+2 & \text{if } \nu = 1. \end{cases}$$

Also

$$\frac{1}{B_k(\lambda)} \leq 1, \ \frac{1}{B_k(\mu)} \leq 1.$$

Thus it follows that the right hand side of (2.13) is an increasing function of $k \geq 1$. Setting $k = 1$ in (2.13), we have

$$\begin{aligned} \rho &\leq 1 - \frac{2\beta\gamma(1-\alpha)(1-\delta)B_1(\nu)}{(1+2\beta\gamma)B_1(\lambda)B_1(\mu)} \\ &= \frac{(1+2\beta\gamma-\beta)(\lambda+1)(\lambda+2)(\mu+1)(\mu+2) - 4\beta\gamma(1-\alpha)(1-\delta)(\nu+1)(\nu+2)}{(1+2\beta\gamma-\beta)(\lambda+1)(\lambda+2)(\mu+1)(\mu+2)} \end{aligned}$$

This completes the proof of Theorem 1. The result is sharp for the functions

$$f(z) = \frac{1}{z} + \frac{4\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)(\lambda+1)(\lambda+2)} z \in \Sigma_p(\lambda, \alpha, \beta, \gamma)$$

and

$$g(z) = \frac{1}{z} + \frac{4\beta\gamma(1-\delta)}{(1+2\beta\gamma-\beta)(\mu+1)(\mu+2)} z \in \Sigma_p(\mu, \delta, \beta\gamma)$$

Taking $\lambda = \mu = \nu = 0$ and $\alpha = \delta$ in Theorem 1, we have

Corollary 2. *If $f, g \in \Sigma_p(0, \alpha, \beta, \gamma)$, then $f * g \in \Sigma_p(0, 1 - \frac{2\beta\gamma(1-\alpha)^2}{1+2\beta\gamma-\beta}, \beta, \gamma)$.*

Putting $\lambda = \mu = \nu = 0$ in Theorem 1, we obtain

Corollary 3. *If $f \in \Sigma_p(0, \alpha, \beta, \gamma)$ and $g \in \Sigma_p(0, \delta, \beta, \gamma)(0 \leq \alpha, \delta < 1)$, then $f * g \in \Sigma_p(0, 1 - \frac{2\beta\gamma(1-\alpha)(1-\delta)}{1+2\beta\gamma-\beta}, \beta, \gamma)$.*

For the next theorem (Theorem 2) and its corollary, we assume that f is given by (1.5), g by (2.5) and h by

$$h(z) = \frac{1}{z} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2) z^k \quad (z \in E). \tag{2.14}$$

Theorem 2. *Let $\lambda \geq 0$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$ and $\frac{1}{2} \leq \gamma \leq 1$. Then*

$$f, g \in \Sigma_p(\lambda, \alpha, \beta, \gamma) \Longrightarrow \frac{1}{2} h \in \Sigma_p(\lambda, \eta, \beta, \gamma),$$

where

$$\eta = 1 - \frac{4\beta\gamma(1-\alpha)^2}{(\lambda+1)(\lambda+2)(1+2\beta\gamma-\beta)}.$$

The result is sharp.

Proof. Since $f \in \Sigma_p(\lambda, \alpha, \beta, \gamma)$, Lemma 1 yields

$$\sum_{k=1}^{\infty} \left(B_k(\lambda) \frac{k(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)} \right)^2 a_k^2 \leq \left(\sum_{k=1}^{\infty} B_k(\lambda) \frac{k(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)} a_k \right)^2 \leq 1.$$

Similarly,

$$\sum_{k=1}^{\infty} \left(B_k(\lambda) \frac{k(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)} \right)^2 b_k^2 \leq 1.$$

Hence

$$\sum_{k=1}^{\infty} \frac{1}{2} \left(B_k(\lambda) \frac{k(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)} \right)^2 (a_k^2 + b_k^2) \leq 1. \tag{2.15}$$

We now want to find the largest $\eta = \eta(\alpha, \beta, \gamma, \lambda)$ such that

$$\sum_{k=1}^{\infty} \frac{1}{2} \left(B_k(\lambda) \frac{k(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\eta)} \right) (a_k^2 + b_k^2) \leq 1. \tag{2.16}$$

Thus (2.15) implies (2.16) if

$$\frac{B_k(\lambda) k(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\eta)} \leq \left(\frac{B_k(\lambda) k(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)} \right)^2,$$

or equivalently,

$$\eta \leq 1 - \frac{2\beta\gamma(1-\alpha)^2}{B_k(\lambda)^k(1+2\beta\gamma-\beta)}. \tag{2.17}$$

Since the right-hand side of (2.17) is an increasing function of $k (k \geq 1)$, we have

$$\eta \leq 1 - \frac{4\beta\gamma(1-\alpha)^2}{(\lambda+1)(\lambda+2)(1+2\beta\gamma-\beta)}.$$

The result is sharp for the functions

$$f(z) = g(z) = \frac{1}{z} + \frac{4\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)(\lambda+1)(\lambda+2)} z. \tag{2.18}$$

For a function f of the form (1.1), it is easy to see that the condition (2.1) provides only a sufficient condition for the function to be the class $\Sigma(\lambda, \alpha, \beta, \gamma)$. More precisely, the function $f \in \Sigma(\lambda, \alpha, \beta, \gamma)$ if

$$\sum_{k=1}^{\infty} k B_k(\lambda)(1+2\beta\gamma-\beta)|a_k| \leq 2\beta\gamma(1-\alpha). \tag{2.19}$$

As a consequence of (2.19), we have

Theorem 3. *For $\lambda > -1$, $0 \le \alpha < 1$, $0 < \beta \le 1$, $\frac{1}{2} \le \gamma \le 1$ and $a_k \ge 0$ $(k \ge 1)$. let*

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k \in \Sigma_p(\lambda, \alpha, \beta, \gamma), \; g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k$$

for $z \in D$. Then

$$f * g \in \Sigma(\lambda, \alpha, \beta, \gamma) \text{ if } |b_k| \le 1 \quad (k \ge 1)$$

and

$$f * g \in \Sigma_p(\lambda, \alpha, \beta, \gamma) \text{ if } 0 \le b_k \le 1 \quad (k \ge 1).$$

Proof. We observe that

$$\sum_{k=1}^{\infty} kB_k(\lambda)(1 + 2\beta\gamma - \beta)|a_k b_k| = \sum_{k=1}^{\infty} kB_k(\lambda)(1 + 2\beta\gamma - \beta)a_k|b_k|$$
$$\le \sum_{k=1}^{\infty} kB_k(\lambda)(1 + 2\beta\gamma - \beta)a_k \le 2\beta\gamma(1 - \alpha),$$

and so, in view of (2.19), it follows that $f * g \in \Sigma(\lambda, \alpha, \beta, \gamma)$. If $0 \le b_k \le 1$, then the above observation yields the inequality (2.19) which, using Lemma 1, proves that $f * g \in \Sigma_p(\lambda, \alpha, \beta, \gamma)$. This completes the proof of Theorem 3.

Corollary 4. *Let f and g be as defined in Theorem 3. If $f \in \Sigma_p(\lambda, \alpha, \beta, \gamma)$, then*

(*i*) $f * g \in \Sigma(0, \alpha, \beta, \gamma)$ if $|b_k| \le 1 \quad (k \ge 1)$.
(*ii*) $f * g \in \Sigma_p(0, \alpha, \beta, \gamma)$ if $0 \le b_k \le 1 \quad (k \ge 1)$.

3. Order of Starlikeness

Theorem 4. *Let $\lambda \ge 0$, $0 \le \alpha < 1$, $0 < \beta \le 1$ and $\frac{1}{2} \le \gamma \le 1$. If $f \in \Sigma_p(\lambda, \alpha, \beta, \gamma)$, then*

$$f \in \Sigma_p^*\left(\frac{(1 + 2\beta\gamma - \beta)(\lambda + 1)(\lambda + 2) - 4\beta\gamma(1 - \alpha)}{(1 + 2\beta\gamma - \beta)(\lambda + 1)(\lambda + 2) + 4\beta\gamma(1 - \alpha)}\right).$$

The result is sharp.

Proof. In view of Lemma 1 and Theorem C, it suffices to show that, for a function f of the form (1.5), the inequality (2.1) implies that

$$\sum_{k=1}^{\infty} \left(\frac{k + \frac{(1+2\beta\gamma-\beta)(\lambda+1)(\lambda+2)-4\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)(\lambda+1)(\lambda+2)+4\beta\gamma(1-\alpha)}}{1 - \frac{(1+2\beta\gamma-\beta)(\lambda+1)(\lambda+2)-4\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)(\lambda+1)(\lambda+2)+4\beta\gamma(1+\alpha)}}\right) a_k \le 1, \tag{3.1}$$

which holds if

$$\frac{k + \frac{(1+2\beta\gamma-\beta)(\lambda+1)(\lambda+2)-4\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)(\lambda+1)(\lambda+2)+4\beta\gamma(1-\alpha)}}{1 - \frac{(1+2\beta\gamma-\beta)(\lambda+1)(\lambda+2)-4\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)(\lambda+1)(\lambda+2)+4\beta\gamma(1+\alpha)}} \le B_k(\lambda)\frac{k(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)}$$

or equivalently, when

$$G(k,\lambda,\alpha,\beta,\gamma) = \frac{(k+1)(1+2\beta\gamma-\beta)(\lambda+1)(\lambda+2)+4(k-1)\beta\gamma(1-\alpha)}{4kB_k(\lambda)(1+2\beta\gamma-\beta)} \le 1.$$

We note that $G(1,\lambda,\alpha,\beta,\gamma)=1$. Therefore, it suffices to show that $G(k,\lambda,\alpha,\beta,\gamma)$ is a decreasing function of $k(k \ge 1)$, that is, that

$$\begin{aligned}\frac{G(k+1,\lambda,\alpha,\beta,\gamma)}{G(k,\lambda,\alpha,\beta,\gamma)} &= \frac{[(k+2)(1+2\beta\gamma-\beta)(\lambda+1)(\lambda+2)+4k\beta\gamma(1-\alpha)]}{[(k+1)(1+2\beta\gamma-\beta)(\lambda+1)(\lambda+2)+4(k-1)\beta\gamma(1-\alpha)]}\\ &\quad\times\frac{k(k+2)}{(k+1)(\lambda+k+2)} \le 1. \qquad (3.2)\end{aligned}$$

But the inequality (3.2) holds if and only if, for each fixed $\lambda(\lambda \ge 0)$ and $\alpha(0 \le \alpha < 1)$, $\beta(0 < \beta \le 1)$ and $\gamma(\frac{1}{2} \le \gamma \le 1)$, we have

$$\begin{aligned}H(k) &= [(k+1)(1+2\beta\gamma-\beta)(\lambda+1)(\lambda+2)+4(k-1)\beta\gamma(1-\alpha)](k+1)(\lambda+k+2)\\ &\quad-[(k+2)(1+2\beta\gamma-\beta)(\lambda+1)(\lambda+2)+4k\beta\gamma(1-\alpha)]k(k+2) \ge 0.\end{aligned}$$

We observe that $H(1) = (4\lambda+3)(\lambda+1)(\lambda+2)(1+2\beta\gamma-\beta) - 12\beta\gamma(1-\alpha) \ge 0$ and

$$\begin{aligned}H(k+1)-H(k) &= [2(1+2\beta\gamma-\beta)(\lambda+1)(\lambda+2)+8\beta\gamma(1-\alpha)]\lambda k\\ &\quad+(3\lambda+1)(\lambda^2+3\lambda)(1+2\beta\gamma-\beta)+4\beta\gamma((3\lambda+\alpha)+(1-\alpha)\lambda)\\ &\quad+2(1-\beta)(3\lambda+1) \ge 0.\end{aligned}$$

for all $\lambda \ge 0$, $0 \le \alpha < 1$, $0 < \beta \le 1$ and $\frac{1}{2} \le \gamma \le 1$. This completes the proof of Theorem 4. The result is sharp for the functions

$$f(z) = g(z) = \frac{1}{z} + \frac{4\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)(\lambda+1)(\lambda+2)}z.$$

By a similar proof of Theorem 4, we can obtain the following theorem.

Theorem 5. *Let f and g be as defined in (1.5) and (2.5), respectively. Then, for $\lambda \ge 0$, $0 \le \alpha < 1$, $0 < \beta \le 1$ and $\frac{1}{2} \le \gamma \le 1$,*

$$f,g \in \Sigma_p(\lambda,\alpha,\beta,\gamma) \Longrightarrow \frac{1}{2}h \in \Sigma_p^*\left(\frac{(\lambda+1)^2(\lambda+2)^2(1+2\beta\gamma-\beta)^2-16(\beta\gamma(1-\alpha))^2}{(\lambda+1)^2(\lambda+2)^2(1+2\beta\gamma-\beta)^2+16(\beta\gamma(1-\alpha))^2}\right),$$

where h is given by (2.14). The result is sharp for the function given by (2.18).

4. Integral Transforms

Theorem 6. *Let f be of the form (1.5) and $\lambda \geq 0$. If $f \in \Sigma_p(\lambda, \alpha, \beta, \gamma)$, then the integral transforms*

$$F_c(z) = c \int_0^1 u^c f(uz)du \ (0 < c < \infty)$$

are in the class $\Sigma_p(\lambda, \frac{2+c\alpha}{2+c}, \beta, \gamma)$. the result is sharp.

Proof. From the definition of F_c, we have

$$F_c = c \int_0^1 u^c f(uz)du = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{ca_k}{k+c+1} z^k.$$

In view of Lemma 1, it is sufficient to show that

$$\sum_{k=1}^{\infty} B_k(\lambda) \frac{k(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\frac{2+c\alpha}{2+c})} \frac{ca_k}{k+c+1} \leq 1. \tag{4.1}$$

Since $f \in \Sigma_p(\lambda, \alpha, \beta, \gamma)$, (4.1) will be satisfied if,

$$\frac{c}{(1-\frac{2+c\alpha}{2+c})(k+c+1)} \leq \frac{1}{1-\alpha}.$$

or equivalently, when

$$I(k, \lambda, \alpha, c) = \frac{(1-\alpha)c}{(1-\frac{2+c\alpha}{2+c})(k+c+1)} \leq 1.$$

Since $I(k, \lambda, \alpha, c)$ is a decreasing function of $k(k \geq 1)$, our proof is completed. The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{2\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)(\lambda+1)(\lambda+2)} z.$$

5. Extreme Points and Its Application

In view of Lemma 1, we observe that $\Sigma_p(\lambda, \alpha, \beta, \gamma)$ is a closed convex family. Using Theorem B (or Lemma 1), we may obtain the extreme points of $\Sigma_p(\lambda, \alpha, \beta, \gamma)$.

Theorem 7. *The extreme points of $\Sigma_p(\lambda, \alpha, \beta, \gamma)$, where $\lambda > -1$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$ and $\frac{1}{2} \leq \gamma \leq 1$, are the functions given by*

$$F_0(z) = \frac{1}{z}, \ F_k(z) = \frac{1}{z} + \frac{2\beta\gamma(1-\alpha)}{kB_k(\lambda)(1+2\beta\gamma-\beta)} z^k \ (k = 1, 2, \ldots), \tag{5.1}$$

where $B_k(\lambda)$ is defined in (2.2).

Proof. Since the operator $D^\lambda : \Sigma_p(\lambda,\alpha,\beta,\gamma) \to \Sigma_p(0,\alpha,\beta,\gamma)$ is an isomorphism from $\Sigma_p(\lambda,\alpha,\beta,\gamma)$ onto $\Sigma_p(0,\alpha,\beta,\gamma)$, it preserves extreme points. In view of (2.3) and Theorem B, it follows that the extreme points of $\Sigma_p(\lambda,\alpha,\beta,\gamma)$ are given by (5.1).

Corollary 5. *Let f be of the form (1.5), and $\lambda \geq 0$. If $f \in \Sigma_p(\lambda,\alpha,\beta,\gamma)$ then*

$$\frac{1}{r} - \frac{4\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)(\lambda+1)(\lambda+2)} r \leq |f(z)| \leq \frac{1}{r} + \frac{4\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)(\lambda+1)(\lambda+2)} r \, (|z|=r),$$

with equality for the function $F_1(z)$ defined in (5.1).

Proof. As a consequence of Theorem 7, we have

$$\frac{1}{r} - \max_k M(\lambda,\alpha,\beta,\gamma,k) r^k \leq |f(z)| \leq \frac{1}{r} + \max_k M(\lambda,\alpha,\beta,\gamma,k) r^k,$$

where

$$M(\lambda,\alpha,\beta,\gamma,k) = \frac{2\beta\gamma(1-\alpha)}{kB_k(\lambda)(1+2\beta\gamma-\beta)}.$$

It suffices to verify that $M(\lambda,\alpha,\beta,k)$ is a decreasing function of $k (k \geq 1)$, that is, that

$$\frac{M(\lambda,\alpha,\beta,\gamma,k+1)}{M(\lambda,\alpha,\beta,\gamma,k)} = \frac{k(k+2)}{(k+1)(\lambda+k+2)} \leq 1,$$

which proves the result.

Remark. For $\lambda = 0$, $\beta = 1$, $\gamma = 1$ and $\lambda = 0$, Theorem 7 and Corollary 5 above give the corresponding results for $\Sigma_p(0,\alpha,1,1)$ and $\Sigma_p(0,\alpha,\beta,\gamma)$, respectively.

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