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PRIME NONASSOCIATIVE RINGS WITH SKEW DERIVATIONS

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Abstract. Let R be a prime nonassociative ring, G the nucleus of R and s, t be automorphisms of R.

(I) Suppose that δ is an s-derivation of R such that $s\delta = \delta s$ and λ is an t-derivation of R. If $\lambda\delta^n = 0$ and $\delta^n(R) \subseteq G$, where n is a fixed positive integer, then $\lambda = 0$ or $\delta^{3n-1} = 0$.

(II) Assume that δ and λ are derivations of R. If there exists a fixed positive integer n such that $\lambda^n\delta = 0$, and $\delta(R) \subseteq G$ or $\lambda^n(R) \subseteq G$, then $\delta^2 = 0$ or $\lambda^{6n-4} = 0$.

1. Introduction

Let R be a nonassociative ring. We adopt the usual notation for commutators and associators: $[x, y] = xy - yx$ and $(x, y, z) = (xy)z - x(yz)$ for $x, y, z \in R$. We shall denote the nucleus of R by G. Thus G consists of all elements n in R such that $(n, R, R) = (R, n, R) = (R, R, n) = 0$. Denote the group of all automorphsisms of R by $Aut(R)$. An additive mapping δ from R into R is called a skew derivation or an s-derivation if $\delta(xy) = \delta(x)y + s(x)\delta(y)$ for all x, y in R, where $s \in Aut(R)$. If s is the identity automorphism of R then δ is called a derivation of R. Let $Der(R)$ be the Lie ring of derivations of R. A ring R is called prime if the product of any two nonzero ideals of R is nonzero.

Posner [3] proved that if R is a prime associative ring of characteristic not two with derivations λ and δ then $\lambda\delta \in Der(R)$ implies $\lambda = 0$ or $\delta = 0$. Jensen [2] partially extended this result. Two of his results are as follows: If R is a prime associative ring with derivations λ, δ and there exists a fixed positive integer n such that $\lambda\delta^n = 0 (\lambda^n\delta = 0)$ then $\lambda = 0$ or $\delta^{4n-1} = 0$ ($\delta^2 = 0$ or $\lambda^{12n-9} = 0$). In this paper, we improve and generalize these results to the prime nonassociative rings with skew derivations.

In every ring R we have the Teichmüller identity

$$(\omega x, y, z) - (\omega, xy, z) + (\omega, x, yz) = \omega(x, y, z) + (\omega, x, y)z \quad \text{for all} \quad \omega, x, y, z \in R. \quad (1)$$

Note that the associator (x, y, z) is linear in each argument. Thus using (1), we have that G is an associative subring of R.

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2. Results

Our first main result is the following Theorem 1, which improves and generalizes Jensen's Theorem 1.

Theorem 1. [5] *Let R be a prime nonassociative ring and let $s, t \in Aut(R)$. Suppose that δ is an s-derivation of R such that $s\delta = \delta s$ and λ is an t-derivation of R. If $\lambda\delta^n = 0$ and $\delta^n(R) \subseteq G$, where n is a fixed positive integer, then $\lambda = 0$ or $\delta^{3n-1} = 0$.*

To prove Theorem 1 we need a Lemma. The proof of the Lemma is the same as that of [4, lemma 1], except that the equation (4) of [4] is replaced by

$$\delta^n(xy) = \sum_{i=0}^{n} d^i(s^{n-i}(x))d^{n-i}(y) \in A.$$

Lemma. *Let A be subring of R. If $\delta^n(R) \subseteq A$ then $R\delta^{3n-1}(R) \subseteq A$. Proof of Theorem 1.*

Let $\ker(\lambda) = \{c \in R : \lambda(c) = 0\}$. Then $\ker(\lambda)$ is the subring of constants of R under λ. The hypothesis $\lambda\delta^n = 0$ implies $\delta^n(R) \subseteq \ker(\lambda)$. By the Lemma, we get $R\delta^{3n-1}(R) \subseteq \ker(\lambda)$. Since $\delta^n(R) \subseteq G$ and $s\delta = \delta s$, using [4, Theorem] we obtain that R is associative or $\delta^{3n-1} = 0$. Assume that $\delta^{3n-1} \neq 0$. Then R is associative. Because of $R\delta^{3n-1}(R) \subseteq \ker(\lambda)$ and $\lambda\delta^n = 0$, for all x, y, z in R we have

$$0 = \lambda(x\delta^{3n-1}(y)) = \lambda(x)\delta^{3n-1}(y) + t(x)\lambda(\delta^{3n-1}(y)) = \lambda(x)\delta^{3n-1}(y)$$

and so $\lambda(zx)\delta^{3n-1}(y) = 0$. The last two equalities imply

$$\begin{aligned} 0 &= (\lambda(z)x + t(z)\lambda(x))\delta^{3n-1}(y) \\ &= \lambda(z)x\delta^{3n-1}(y) + t(z)\lambda(x)\delta^{3n-1}(y) \\ &= \lambda(z)x\delta^{3n-1}(y). \end{aligned}$$

By the primeness of R, this implies $\lambda(z) = 0$ or $\delta^{3n-1}(y) = 0$. In view of $\delta^{3n-1} \neq 0$, we obtain $\lambda(z) = 0$ for all z in R. Thus $\lambda = 0$, as desired.

Our second main result is the following Theorem 2, which improves and generalizes Jensen's Theorem 2.

Theorem 2. *Let R be a prime nonassociative ring and let δ and λ be derivations of R. If there exists a fixed positive integer n such that $\lambda^n\delta = 0$, and $\delta(R) \subseteq G$ or $\lambda^n(R) \subseteq G$, then $\delta^2 = 0$ or $\lambda^{6n-4} = 0$.*

Proof. The derivations of R form a Lie ring under commutation. Therefore $[\delta, \lambda] = \delta\lambda - \lambda\delta$ is a derivation, $[\delta\lambda - \lambda\delta, \lambda] = \delta\lambda^2 - 2\lambda\delta\lambda + \lambda^2\delta$ is a derivation, and $[\delta\lambda^2 - 2\lambda\delta\lambda +$

$\lambda^2\delta,\lambda] = \delta\lambda^3 - 3\lambda\delta\lambda^2 + 3\lambda^2\delta\lambda - \lambda^3\delta$ is also a derivation. Continuing we may conclude that

$$\sum_{i=0}^{2n-1} \binom{2n-1}{i}(-1)^i\lambda^i\delta\lambda^{2n-1-i}$$

is a derivation. The coefficients are not germane to the rest of the proof, so we suppress them from here on out. Thus, using $\lambda^n\delta = 0$ we have that

$$\delta\lambda^{2n-1} + \lambda\delta\lambda^{2n-2} + \cdots + \lambda^{n-1}\delta\lambda^n \text{ is a derivation of } R. \tag{2}$$

Since $\delta(R) \subseteq G$ or $\lambda^n(R) \subseteq G$, applying [4, Theorem] we obtain that R is associative, or $\delta^2 = 0$ or $\lambda^{3n-1} = 0$. If $\delta^2 = 0$ or $\lambda^{3n-1} = 0$, then we are done.

Suppose that $\delta^2 \neq 0$ and $\lambda^{3n-1} \neq 0$. Then R is associative. Because of $\lambda^n\delta = 0$, we get $(\delta\lambda^{2n-1} + \lambda\delta\lambda^{2n-1} + \cdots + \lambda^{n-1}\delta\lambda^n)\delta = 0$. In view of $\delta^2 \neq 0$, by (2) and Theorem 1 the last equality implies

$$\delta\lambda^{2n-1} + \lambda\delta\lambda^{2n-2} + \cdots + \lambda^{n-1}\delta\lambda^n = 0. \tag{3}$$

Premultiplying (3) by λ^{n-1} and applying $\lambda^n\delta = 0$, we obtain

$$\lambda^{n-1}\delta\lambda^{2n-1} = 0. \tag{4}$$

Using (4) and premultiplying (3) by λ^{n-2}, it follows that $\lambda^{n-2}\delta\lambda^{2n-1} + \lambda^{n-1}\delta\lambda^{2n-2} = 0$. Hence, we have $0 = (\lambda^{n-2}\delta\lambda^{2n-1} + \lambda^{n-1}\delta\lambda^{2n-2})\lambda$ and so by (4)

$$\lambda^{n-2}\delta\lambda^{2n} = 0. \tag{5}$$

As the proof of [2, Theorem 2], we obtain $\lambda^{n-3}\delta\lambda^{2n+1} = \lambda^{n-4}\delta\lambda^{2n+2} = \cdots = \delta\lambda^{3n-2} = 0$. Combining (4) with (5) yields

$$\lambda^{n-2}[\delta,\lambda]\lambda^{2n-1} = 0. \tag{6}$$

Since $\lambda^n\delta = 0$, we get $\lambda^n[\delta,\lambda] = 0$. Thus, replacing δ by $[\delta,\lambda]$ and comparing (4), (5) and (6), and as the last proof we have

$$\lambda^{n-3}[[\delta,\lambda],\lambda]\lambda^{2n-1} = 0. \tag{7}$$

Continuing in this manner, we finally obtain

$$\mu\lambda^{2n-1} = 0, \text{ where } \mu = [[\cdots[[\delta,\underbrace{\lambda],\lambda],\cdots],\lambda]}_{(n-1)'s\ \lambda}. \tag{8}$$

Because of μ is a derivation of R, by (8) and Theorem 1, we get $\mu = 0$ or $\lambda^{6n-4} = \lambda^{3(2n-1)-1} = 0$. If $\lambda^{6n-4} = 0$, then we are done. Assume that $\mu = 0$. Thus, as the beginning of the proof, we may suppose that

$$\nu = \delta\lambda^{n-1} + \lambda\delta\lambda^{n-2} + \cdots + \lambda^{n-1}\delta = 0. \tag{9}$$

Using (9) repeatedly and $\lambda^n\delta = 0$, we have $0 = \lambda^{n-1}\nu = \lambda^{n-1}\delta\lambda^{n-1}$, $0 = \lambda^{n-2}\nu\lambda = \lambda^{n-2}\delta\lambda^n, \ldots$, and finally we obtain

$$\delta\lambda^{2n-1} = 0. \tag{10}$$

By Theorem 1 and $\delta \neq 0$, (10) implies $\lambda^{6n-4} = \lambda^{3(2n-1)-1} = 0$, as desired.

Chung and Luh [1] showed that in a prime associative ring with characteristic 2, the nilpotency of nilpotent derivation must be of the form 2^k, where $k \in \mathbb{N}$. Therefore, when R is not 2-torsion free, the possible values for nilpotency in Theorem 1 and Theorem 2 are further limited. For example, if we assume in Theorem 1 or Theorem 2 that the characteristic of R is 2, $\delta\lambda^{21} = 0$ and $\delta \neq 0$, or $\lambda^{11}\delta = 0$ and $\delta^2 \neq 0$, then the nilpotency of λ must be 1, 2, 4, 8, 16, or 32.

References

[1] L. O. Chung and J. Luh, "Nilpotency of derivations II," *Proc. Amer. Math. Soc.*, 91(1984), 357-358.

[2] D. W. Jensen, "Nilpotency of derivations in prime rings," *Proc. Amer. Math. Soc.*, 123(1995), 2633-2636.

[3] E. C. Posner, "Derivations in prime rings," *Proc. Amer. Math. Soc.*, 8(1957), 1093-1100.

[4] C. T. Yen, "Prime ring with a derivation whose some power image is contained in the nucleus," *Soochow J. Math.*, 21(1995), 475-478.

[5] C. T. Yen, Nonassociative rings with skew derivations and rings with associators in the nuclei, Ph. D. thesis, Taiwan University, 1995.

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