TAMKANG JOURNAL OF MATHEMATICS
Volume 28, Number 4, Winter 1997

COMPARISON THEOREMS FOR A NONLINEAR THREE-TERM EQUATION

SHU TANG LIU AND SUI SUN CHENG

Abstract. It is well known that qualitative properties of special classes of difference equations can be inferred from their associated "characteristic equations." By means of limit type comparison theorems, we may then deduce qualitative properties for related but perhaps more general classes of difference equations. In this paper, positive solutions of a class of nonlinear difference equations are shown to exist by means of such a scheme.

1. Introduction

Qualitative theory of nonlinear discrete-time dynamical systems of the form $x_{n+1} = f(x_n)$ has been investigated to a great extent in recent years. Relatively little is known, however, for nonlinear equations of the form

$$x_{n+1} = F_n(x_n, x_{n-\sigma}).$$

There do exist, among others, several interesting articles [1] which are concerned with asymptotic behaviour of such equations. Since most of these equations arise from population models [1, pp. 337-349], efforts in such areas are therefore warranted.

In this paper, we will be concerned with a class of nonlinear discrete time dynamical system of the form

$$x_{n+1} = \alpha x_n - \beta_n f(x_{n-\sigma}^{\tau} x_n^{1-\tau}), \quad n = 0, 1, 2, \ldots, \tag{1}$$

where α is a real number, $\{\beta_n\}_{n=0}^{\infty}$ is a positive sequence, σ is a nonnegative integer, τ is a positive integer, and f is a real function such that $f(x) > 0$ for $x > 0$.

Two particular cases of (1) will play important roles in the sequel. First of all, when $f(x) = x$ for $-\infty < x < \infty$, equation (1) reduces to the equation

$$x_{n+1} = \alpha x_n - \beta_n x_{n-\sigma}^{\tau} x_n^{1-\tau}, \tag{2}$$

which yields the following well known equation

$$x_{n+1} - x_n + \beta_n x_{n-\sigma} = 0$$

Received February 21, 1997.
1991 *Mathematics Subject Classification.* 39A10
Key words and phrases. Nonlinear three-term equation, difference equation, characteristic equation.

in case α and τ are also equal to 1. Next, when $\beta_n = \beta > 0$ for $n \geq 0$, then (2) becomes

$$x_{n+1} = \alpha x_n - \beta x_{n-\sigma}^{\tau} x_n^{1-\tau}. \tag{3}$$

Since such an equation has constant coefficients, solutions of the form $\{\lambda^n\}$ may be found by direct substitution. Such a scheme will be carried out in the sequel.

We will be concerned with the existence of eventually positive sequences which satisfies (1) for all large n. Such sequences will be called eventually positive solutions of (1). It is not clear that equation (1) may possess eventually positive solutions. On the other hand, it seems rather easy to generate solutions of (1) by imposing initial values and then calculate the subsequent terms by means of equation (1) itself. Note however that such a scheme may fail if one of the terms, say x_n, is equal to zero, for then $x_n^{1-\tau}$ may become infinite. In any case, by means of several comparison theorems, we will find sufficient conditions in terms of the parameters α, τ, σ, $\{\beta_n\}$ and the function f so that eventually positive solutions will exist.

2. Relation with a Linear Equation

For the sake of definiteness, let us first recall the following assumptions: α is a real number, β a positive number, σ a nonnegative integer, τ a positive integer, $\{\beta_n\}$ a positive sequence and f a real function such that $f(x) > 0$ for $x > 0$. We need to make several additional observations. If $\{x_n\}$ is an eventually positive solution of (1), then

$$x_{n+1} = \alpha x_n - \beta_n f(x_{n-\sigma}^{\tau} x_n^{1-\tau}) \leq \alpha x_n$$

for all large n. Thus α must be positive. Furthermore, $\{x_n\}$ is decreasing when $\alpha \leq 1$ and strictly decreasing to zero when $\alpha < 1$. Next, note that an eventually positive solution of (3) may take the form $\{\lambda^n\}$, where λ is a positive number. For such a solution to exist, it suffices to solve the positive roots of the following "characteristic equation"

$$\Gamma(\lambda) \equiv \lambda - \alpha + \beta\lambda^{-\sigma\tau} = 0 \tag{4}$$

obtained by substituting $\{\lambda^n\}$ into equation (3).

The following result is clear.

Lemma 1. *If equation* (1) *has an eventually positive solution, then* $\alpha > 0$. *Every eventually positive solution of* (1) *is eventually decreasing when* $\alpha \in (0, 1]$ *and eventually strictly decreasing when* $\alpha \in (0, 1)$.

In particular, if $\{x_n\}$ is an eventually positive solution of (3), then

$$x_n < \alpha x_{n-1} < \alpha^2 x_{n-2} < \cdots < \alpha^{\sigma} x_{n-\sigma}, \qquad \sigma > 0$$

and

$$x_{n-\sigma}^{\tau} > \alpha^{-\sigma\tau} x_n, \qquad \sigma > 0 \tag{5}$$

for all large n.

Lemma 2. *If the characteristic equation* (4) *has a positive root* λ, *then* (3) *has an eventually positive solution of the form* $\{\lambda^n\}$. *Conversely, if* (3) *has an eventually positive solution, then* (4) *has positive root.*

Proof. To see that the converse holds, assume that $\{x_n\}$ is an eventually positive solution of (3). Consider first the case where $\sigma = 0$. Then

$$\Gamma(\lambda) = \lambda - \alpha + \beta$$

has a root $\lambda_0 = \alpha - \beta$. This root is positive, since

$$0 < \frac{x_{n+1}}{x_n} = \alpha - \beta$$

in view of (3). Next we consider the case where $\sigma > 0$. Assume to the contrary that $\Gamma(\lambda)$ does not have any positive roots. Since $\Gamma(0^+) = +\infty$ and $\Gamma(+\infty) = +\infty$, thus there is some positive number μ such that $\Gamma(\lambda) \geq \mu > 0$ for all $\lambda > 0$. Let

$$\Omega = \{\lambda \in (-\infty, \infty) | x_{n+1} \leq \lambda x_n \text{ for all large } n\}.$$

Since $x_{n+1} < \alpha x_n$ for all large n, α belongs Ω.

Let $\lambda_0 = \alpha$. In view of (3) and (5),

$$x_{n+1} = \alpha x_n - \beta x_{n-\sigma}^{\tau} x_n^{1-\tau} \leq \alpha x_n - \beta \alpha^{-\sigma\tau} x_n = \alpha x_n - \Gamma(\alpha) x_n \leq (\alpha - \mu) x_n$$

for all large n. This shows that $\lambda_1 = \alpha - \mu \in \Omega$. By induction, it is not difficult to see that $\alpha - t\mu \in \Omega$ for all $t = 0, 1, 2, \ldots$ This, clearly is a contradiction. The proof is complete.

An interesting and novel application of the above Lemma can be obtained by noting that equation (4) is also the characteristic equation of the linear difference equation

$$x_{n+1} - \alpha x_n + \beta x_{n-\sigma\tau} = 0. \tag{6}$$

It is well known that equation (6) has an eventually positive solution if, and only if, (4) has a positive root. We may thus conclude that equation (3) has an eventually positive solution if, and only if, (6) has an eventually positive solution.

Theorem 1. *The nonlinear equation* (3) *has an eventually positive solution if, and only if, the linear equation* (6) *has an eventually positive solution.*

Oscillation criteria for difference equations of the form (6) are known. Indeed, when $\sigma = 0$, it is easy to see that (6) has an eventually positive solution if, and only if, $\alpha > \beta$. When $\tau\sigma > 0$, it is known [2, Theorem 2.2] that (6) has an eventually positive solution if, and only if, the condition

$$(\beta \leq 0 \text{ or } \alpha > 0) \quad \text{and} \quad \left(\beta \leq 0 \text{ or } \beta \leq \alpha^{\sigma\tau+1} \frac{(\sigma\tau)^{\sigma\tau}}{(\sigma\tau+1)^{\sigma\tau+1}}\right)$$

holds.

3. Comparison Theorems

We now proceed to find relations between equation (3) or (6) and the nonlinear equation (1).

Lemma 3. *Suppose $\beta \neq \alpha$, $\alpha < 1$ and*

$$\liminf_{n\to\infty} \beta_n = \beta > 0. \tag{7}$$

If there is an eventually positive sequence $u = \{u_n\}$ which satisfies

$$u_{n+1} \leq \alpha u_n - \beta_n u_{n-\sigma}^{\tau} u_n^{1-\tau}, \quad n = 0, 1, 2, \ldots, \tag{8}$$

then (3) has an eventually positive solution.

Proof. Consider first the case $\sigma = 0$. Note that

$$0 < u_{n+1} \leq \alpha u_n - \beta_n u_n = (\alpha - \beta_n) u_n$$

for all large n. Thus $\beta_n \leq \alpha$ for all large n. In view of (7) and our assumption that $\beta \neq \alpha$, we see further that $0 < \beta < \alpha$. But then $\lambda = \alpha - \beta$ is a positive solution of (4). In view of Lemma 2, we see that (3) has an eventually positive solution.

Next, assume that $\sigma > 0$. Note that $\Gamma(\lambda)$ tends to positive infinity as λ tends to zero from the right. We assert further that $\Gamma(\mu) \leq 0$ for some positive number μ, so that $\Gamma(\lambda)$ will have a positive root in $(0, \mu]$. To see this, let

$$\omega_n = \frac{u_n}{u_{n-1}},$$

then by means of the same reasoning used in obtaining Lemma 1, $0 < \omega_n < 1$ for all large n. Furthermore, in view of (8),

$$\omega_{n+1} = \frac{u_{n+1}}{u_n} \leq \alpha - \beta_n \frac{u_{n-\sigma}^{\tau}}{u_n^{\tau}} = \alpha - \beta_n \left(\Pi_{j=0}^{\sigma-1} \frac{1}{\omega_{n-j}} \right)^{\tau}. \tag{9}$$

Let $\omega = \limsup_{n\to\infty} \omega_n$. Then $\omega \in [0, 1]$. However, ω cannot equal to zero since in view of (9)

$$\beta_n \left(\Pi_{j=0}^{\sigma-1} \frac{1}{\omega_{n-j}} \right)^{\tau} \leq \alpha.$$

We assert further that $\Gamma(\omega) \leq 0$. Indeed, for any small positive number ϵ, we have

$$\omega_n \leq (1 + \epsilon)\omega$$

for all large n; and by (7), we have

$$\beta_n \geq (1 - \epsilon)\beta$$

for all large n. Thus we may infer from (9) that

$$\omega_{n+1} \leq \alpha - (1-\epsilon)\beta\omega^{-\sigma\tau}(1+\epsilon)^{-\sigma\tau}$$

for all large n. By taking the superior limits of both sides of the above inequality, we arrive at

$$\omega \leq \alpha - (1-\epsilon)\beta\omega^{-\sigma\tau}(1+\epsilon)^{-\sigma\tau},$$

which holds for every small ϵ. This verifies our assertion and also concludes our proof.

In case $\beta_n = \beta > 0$ for $n \geq 0$, we may now see that the following result holds.

Theorem 2. *Suppose $0 < \beta < \alpha < 1$. Then equation (3) has an eventually positive solution if, and only if, there is an eventually positive sequence which satisfies*

$$u_{n+1} \leq \alpha u_n - \beta u_{u-\sigma}^{\tau} u_n^{1-\tau} \tag{10}$$

for all large n.

As another application, assume that $0 < \beta < \alpha < 1$, and that

$$\liminf_{x \to 0^+} \frac{f(x)}{x} \geq 1. \tag{11}$$

If equation (1) has an eventually positive solution $\{x_n\}$, then

$$0 = x_{n+1} - \alpha x_n + \beta \frac{f(x_{n-\sigma}^{\tau} x_n^{1-\tau})}{x_{n-\sigma}^{\tau} x_n^{1-\tau}} x_{n-\sigma}^{\tau} x_n^{1-\tau}$$

for all large n. Furthermore, since $\{x_n\}$ decreases to zero,

$$\liminf_{n \to \infty} \beta \frac{f(x_{n-\sigma}^{\tau} x_n^{1-\tau})}{x_{n-\sigma}^{\tau} x_n^{1-\tau}} \geq \beta.$$

By means of Lemma 3, we thus see that equation (3) has an eventually positive solution.

Theorem 3. *Suppose that $0 < \beta < \alpha < 1$ and that (11) holds. If equation (1) has an eventually positive solution, then so does equation (3).*

A partial converse of Theorem 3 also holds. To see this, we will need the following Lemma.

Lemma 4. *Suppose $0 < \beta < \alpha < 1$. If λ_0 is a positive root of $\Gamma(\lambda)$, then for any positive integer N and positive number δ, the sequence $\{v_n\}$ defined by*

$$v_{n+1} \geq \alpha v_n - \beta v_{n-\sigma}^{\tau} v_n^{1-\tau}, n = 0, 1, \ldots, N-1, \tag{12}$$

and

$$v_n = \delta\lambda_0^n, n = -\sigma, -\sigma+1, \ldots, 1, 0,$$

will satisfy

$$v_n \geq \delta \lambda_0^n$$

for $n = 1, 2, \ldots, N.$

Proof. Suppose $\sigma = 0$. Then in view of (4), $\lambda_0 = \alpha - \beta > 0$ and our Theorem is clearly true. Next suppose $\sigma > 0$. Let

$$\omega_n = \frac{v_n}{v_{n-1}}, n = -\sigma + 1, \ldots, 1, 0.$$

Then $\omega_n > 0$ for $-\sigma + 1 \leq n \leq 0$. Furthermore, in view of (12), we have

$$0 \leq \omega_1 - \alpha + \beta\left(\Pi_{j=0}^{\sigma-1} \frac{1}{\omega_{-j}}\right)^\tau = \omega_1 - \alpha + \beta \lambda_0^{-\tau\sigma} = \omega_1 - \lambda_0,$$

which shows that $v_1 \geq \delta\lambda_0$. By induction, it is easy to see that $\omega_n \geq \lambda_0$ for $n = 2, \ldots, N$. The proof is complete.

Theorem 4. *Suppose that* $0 < \beta < \alpha < 1$ *and that*

$$\lim_{x \to 0+} \frac{f(x)}{x} \leq 1.$$

If equation (3) *has an eventually positive solution, then so does equation* (1).

Proof. If equation (3) has an eventually positive solution, then by means of Lemma 2, $\Gamma(\lambda)$ will have a positive root λ_0. Let $\{x_n\}$ be the formal sequence defined by

$$x_n = \delta\lambda_0^n, n = -\sigma, -\sigma + 1, \ldots, 0,$$

and equation (1). We assert that $x_n > 0$ for $n \geq 1$. Otherwise there would exist a positive integer such that $x_n > 0$ for $1 \leq n \leq N - 1$ and $x_N \leq 0$. In view of (1), we see that

$$0 < x_{N-1} < x_{N-2} < \cdots < x_1 < x_0 = 1,$$

and also

$$x_{n+1} - \alpha x_n + \beta x_{n-\sigma}^\tau x_n^{1-\tau} \geq x_{n+1} - \alpha x_n + \beta f\left(x_{n-\sigma}^\tau x_n^{1-\tau}\right) = 0.$$

But in view of Lemma 4, $x_N \geq \lambda_0^N > 0$, which is a contradiction. The proof is complete.

As a corollary of Lemma 2 and Theorems 3 and 4, we see that if $0 < \beta < \alpha < 1$ and

$$\lim_{x \to 0+} \frac{f(x)}{x} = 1, \tag{13}$$

then equation (1) has an eventually positive solution if, and only if, $\Gamma(\lambda)$ has a positive root.

We remark that when $\sigma \neq 0$, then

$$\min_{\lambda>0} \Gamma(\lambda) = \Gamma\Big((\beta\sigma\tau)^{1/(\sigma\tau+1)}\Big) = (\beta\sigma\tau)^{1/(\sigma\tau+1)}\Big(\frac{1+(\sigma\tau)^{\sigma\tau}}{(\sigma\tau)^{\sigma\tau}}\Big) - \alpha.$$

Thus when $\sigma \neq 0$, $\alpha \in (0,1)$,

$$(\beta\sigma\tau)^{1/(\sigma\tau+1)}\Big(\frac{1+(\sigma\tau)^{\sigma\tau}}{(\sigma\tau)^{\sigma\tau}}\Big) \leq \alpha,$$

and (13) holds, then equation (1) will have an eventually positive solution.

As an example, consider the equation

$$x_{n+1} = \frac{1}{3}x_n - \frac{1}{3\cdot 4^3}(1 + x_{n-1}^2 x_n^{-1})x_{n-1}^2 x_n^{-1}. \tag{14}$$

Since

$$\lim_{u\to 0+} \frac{(1+u)u}{u} = 1$$

and

$$0 < \beta = \frac{1}{3\cdot 4^3} < \frac{1}{3} = \alpha < 1,$$

and $\{(1/4)^n\}$ is an eventually positive solution of

$$x_{n+1} = \frac{1}{3}x_n - \frac{1}{3\cdot 4^3}x_{n-1}^2 x_n^{-1},$$

thus (14) will have an eventually positive solution also. We remark that the solution $\{(1/4)^n\}$ is obtained by solving the characteristic equation

$$\Gamma(\lambda) = \lambda - \frac{1}{3} + \frac{1}{3\cdot 4^3}\lambda^{-2} = 0$$

for a positive root $\lambda = 1/4$, and finally that

$$\Gamma\Big((\beta\sigma\tau)^{1/(\sigma\tau+1)}\Big) = \frac{5}{4^2}(\frac{2}{3})^{1/3} < \frac{1}{3} = \alpha.$$

Acknowledgement

The second author is partially supported by the National Science Council of R. O. C.

References

[1] S. Elaydi, G. Graef, G. Ladas and A. Peterson, "Proceedings of the First International Conference on Difference Equations", *Gordan and Breach Publishers*, 1995.

[2] Y. Z. Lin and S. S. Cheng, "Complete characterizations of a class of oscillatory difference equations", *J. Difference Eq. Appl.*, 2(1996), 301-313.

Department of Mathematics, Binzhou Normal College, Binzhou, Shandong 256604, P. R. China.
Department of Mathematics, Tsing Hua University, Hsinchu, Taiwan 30043, R. O. C.