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A NOTE ON THE GENERALIZATION OF HUA'S INEQUALITY

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Abstract. In this paper we establish Lo-Keng Hua's inequality for linear operators in real inner product spaces. Our result generalizes Hua's inequality in real inner product spaces; obtained recently by S. S. Dragomir and G.-S. Yang.

In his book "Additive Theory of Prime Numbers" [2] Lo-Keng Hua introduced the following inequality:

$$(\delta - \sum_{i=1}^{n} x_i)^2 + \alpha \sum_{i=1}^{n} x_i^2 \geq \frac{\alpha}{n+\alpha}\delta^2 \tag{1}$$

where $\delta > 0$, $\alpha > 0$, $x_i \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$, with equality if and only if

$$x_1 = x_2 = \cdots = x_n = \frac{\delta}{n+\alpha} \tag{2}$$

The inequality has been generalized in various ways since then. Chung-Lie Wang [3] has shown that $p \neq 2$ can be taken as a power and that instead of the sums $\sum_{i=1}^{n} x_i$ the integrals can be taken. S.S. Dragomir and G.-S. Yang extended Hua's inequality in various ways, for $p = 2$, to real inner product spaces (see Theorem 2.1 and Theorem 2.7 in [1]). Here we will remain with $p = 2$. We will show that simple unified approach can be taken to this inequality with the following consequences:
a) it generalizes Hua's inequality considerably,
b) it contains all the results from [1] as special cases,
c) it contains the integral result (for $p = 2$) from [3] as the special case.

Here is the main result:

Theorem. *Suppose that X and Y are two real inner product spaces, with their respective norms denoted by $\| \ \|_X$ and $\| \ \|_Y$. If $A : X \longrightarrow Y$ is a bounded linear operator, with its operator norm*

$$\|A\| = \sup_{\|x\|_X \leq 1} \|Ax\|_Y$$

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and $\alpha > 0$, then, for every $x \in X$ and $y \in Y$,

$$||y - Ax||_Y^2 + \alpha||x||_X^2 \geq \frac{\alpha}{||A||^2 + \alpha}||y||_Y^2 \tag{3}$$

with equality if and only if

$$\left(A \neq 0, Ax = \frac{||A||^2}{||A||^2 + \alpha}y \text{ and } ||Ax||_Y = ||A||||x||_X\right) \text{ or } (A = 0 \text{ and } x = 0). \tag{4}$$

Proof. Notice first that for $A = 0$ we get

$$||y||_Y^2 + \alpha||x||_X^2 \geq ||y||_Y^2$$

which is obviously true, since $\alpha > 0$ and $||x||_X^2 \geq 0$. The equality holds if and only if $x = 0$, which, under $A = 0$, is equivalent to (4).

Suppose now that $||A|| > 0$. Recall that

$$||x||_X \geq \frac{||Ax||_Y}{||A||}$$

with equality if and only if $||Ax||_Y = ||A||||x||_X$. Furthermore, Lemma 2.4 from [1], applied on the inner product space Y, and vectors y and Ax, and the constant $\alpha/||A||^2 > 0$ shows that

$$||y - Ax||_Y^2 + \frac{\alpha}{||A||^2}||Ax||_Y^2 \geq \frac{\alpha}{||A||^2 + \alpha}||y||_Y^2,$$

with equality if and only if

$$Ax = \frac{||A||^2}{||A||^2 + \alpha}y.$$

It follows that

$$||y - Ax||_Y^2 + \alpha||x||_X^2 \geq ||y - Ax||_Y^2 + \frac{\alpha}{||A||^2}||Ax||_Y^2 \geq \frac{\alpha}{||A||^2 + \alpha}||y||_Y^2,$$

with equality if and only if (4) holds.

Remark. Notice that inner product was used only in the application of Lemma 2.4 from [1]; otherwise we could start with any two normed spaces. However, see Remark 2.5 in [1], inner product is needed in Lemma 2.4 only to establish the equality part. Consequently, our inequality (3) is valid in the normal spaces as well, with inner product being necessary for the equality case only. Let us also mention, for reader's convenience, that the proof of Lemma 2.4 follows directly from the triangle inequality, equality part of the Schwarz-Cauchy inequality, and Hua's inequality (1) for $n = 1$.

It is now easy to see that statements b) and c) are correct. In order to obtain Theorem 2.1 from [1], consider X, an inner product space, for Y we have to take $\mathbb{R}$, and A is the linear functional $Ax = (x, a)$, where $a \in X$. Take $y = \delta \in \mathbb{R}$; notice that the requirement $\delta > 0$ is actually not necessary (one can check this easily for the original Hua's inequality). The inequality (3) now becomes inequality (2.1) from [1]. Since $||A|| = ||a||$, and $||Ax||_Y = ||A||||x||_X$ if and only if $x = \lambda a$ (in this choice of A), we obtain Theorem 2.1 from [1] as a special case of our theorem. In particular, this also shows that original Hua's inequality is a special case of our result.

Similarly, if we consider an inner product space X, then X^n is an inner product space. Take $A : X^n \longrightarrow X$ to be $A(x_1, \ldots, x_n) = \sum_{i=1}^n x_i$, and recall that $||(x_1, \ldots, x_n)||^2_{X^n} = \sum_{i=1}^n ||x_i||^2_X$. Since $||A||^2 = n$ and $||A(x_1, \ldots, x_n)||_X = ||A||||(x_1, \ldots, x_n)||_{X^n}$ if and only if $x_1 = \cdots = x_n$ we obtain Theorem 2.7 from [1] as a special case of our theorem.

Finally, by taking $X = L^2[0,T]$, $0 < T < \infty$, $Y = \mathbb{R}$ and $Af = \int_0^T f(t)dt$, we obtain, for the case of $p = 2$ (even more general result than), Theorem 2. from [1].

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