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DISCRETE AND PSEUDO ORTHOGONALITY FOR A CLASS OF GENERALIZED HYPERGEOMETRIC FUNCTIONS

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Abstract. In his Ph. D. thesis [11] Satyanarayana defined and studied the generalized hypergeometric functions $I_n^\alpha(x,w)$ and $H_n^\alpha(x,w)$. In the present paper we consider discrete orthogonality for $I_n^\alpha(x,w)$ and pseudo orthogonality for $H_n^\alpha(x,w)$. We also obtain some interesting applications of the results of our investigation.

1. Introduction and Definitions

In the present paper an effort has been made to define and apply a discrete orthogonality for $I_n^\alpha(x,w)$ and pseudo orthogonality for $H_n^\alpha(x,w)$ (see [11] and [7]). For brevity, the following notations of Milne-Thomson [8] (also see Lahiri and Satyanarayana [2]) have been adopted throughout this paper.

$$\Delta_{x,w}f(x) = \frac{f(x+w)-f(x)}{w}$$
$$x^{[\alpha w]} = x(x-w)(x-2w)\cdots(x-\alpha w+w) \tag{1.1}$$

so that

$$\lim_{w\to o}\Delta_{x,w}f(x) = \frac{d}{dx}f(x)$$
$$\Delta_{x,w}x^{[\alpha w]} = \alpha\cdot x^{[\alpha w-w]} \tag{1.2}$$
$$\Delta_{x,w}^n(u_x\cdot v_x) = \sum_{k=0}^{n}\binom{n}{k}\Delta_{x,w}^{n-k}u_{x+kw}\Delta_{x,w}^k v_x \tag{1.3}$$
$$\Delta_{x,w}^{-1}(u_x\cdot v_x) = \sum_{p=0}^{\infty}(-1)^p\Delta_{x,w}^p\Delta_{x,w}^{-p-1}u_{x+pw} \tag{1.4}$$
$$\Delta_{x,w}\left(\frac{x}{w}\right)_R = \frac{R}{w}\left(\frac{x}{w}+1\right)_{R-1}$$

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$$\Delta_{x,w}(-\frac{x}{w})_R = (-\frac{R}{w})(-\frac{x}{w})_{R-1} \tag{1.5}$$

$$\Delta_{x,w}(1-w)^{\frac{x}{w}} = -(1-w)^{\frac{x}{w}} \tag{1.6}$$

Definition (satyanarayana [11])

$$I_{n;\lambda;\ (b_q)}^{\alpha;\mu;\ (a_p)}(x,w) = \frac{1}{n!(x-\mu w)^{[\alpha w]}}\Delta_{x,w}^{n}\left[(x-\mu w)^{[(\alpha+n)w]}\cdot{}_{p+1}F_q((a_p),-\frac{x}{w}+\lambda;(b_q);w)\right] \tag{1.7}$$

For the definition of ${}_{p+1}F_q$ one can refer to Srivastava and Manocha [13, p.42(3)]. In Satyanarayanana [11], it is proved that

$$I_{n;\lambda;\ (b_q)}^{\alpha;\mu;\ (a_p)}(x,w) = \frac{(1+\alpha)_n}{n!}F_{q:1;0}^{p:2;1}\left[\begin{matrix}(a_p):\frac{x}{w}-\mu+1,-n;\ -\frac{x}{w}+\lambda; \\ (b_q):1+\alpha;\qquad ----;\end{matrix}\ w,w\right] \tag{1.8}$$

where $F_{q:s;v}^{p:r;u}(x,y)$ is a double hypergeometric function given in Srivastava and Karlsson [12, p.27(28)].

And from Lahiri and Satyanarayana [7], we have

$$H_{n;\lambda;\ (b_q)}^{\alpha;\mu;\ (a_p)}(x,w) = \frac{1}{n!(x-\mu w)^{[(\alpha+n)w]}}\Delta_{x,w}^{n}(x-\mu w)^{[\alpha w]} \cdot{}_{p+1}F_q((a_p),-\frac{x}{w}+\lambda;(b_q);w)\Big] \tag{1.9}$$

$$= \frac{(-\alpha)_n w^{-2n}}{n!(\frac{x}{w}-\mu-\alpha+1)_n(-\frac{x}{w}+\mu+\alpha)_n} \cdot F_{q:1;0}^{p:2;1}\left[\begin{matrix}(a_p):-n,\frac{x}{w}-\mu+1;\ -\frac{x}{w}+\lambda; \\ (b_q):1+\alpha-n;\qquad ----;\end{matrix}\ w,w\right] \tag{1.10}$$

The function of (1.8) was studied by Lahiri and Satyanarayana ([2], [3] and [4]). In particular, for $\lambda=0$ and $\mu=1$, we have

$$I_{n;0;(a_p)}^{\alpha;1;(a_p)}(x,w) = (1-w)\frac{x}{w}\cdot J_n^{\alpha}(x,w), \tag{1.11}$$

where $J_n^{\alpha}(x,w)$ is a modified Jacobi polynomial defined by Parihar and Patel [9] (also see Lahiri and Satyanarayana [2, p.164(1.5)].);

$$\lim_{w\to 0} I_{n;\lambda;(a_p)}^{\alpha;\mu;(a_p)}(x,w) = e^{-x}L_n^{\alpha}(x) \tag{1.12}$$

where $L_n^{\alpha}(x)$ is the Laguerre polynomial (See Rainville [10]). the function of (1.9) was studied by Lahiri and Satyanarayana [7], and we have

$$\lim_{w\to 0}\{e^{x}x^{2n}H_{n;\lambda;(b_q)}^{\alpha;\mu;(a_p)}(x,w)\} = L_n^{\alpha-n}(x), \tag{1.13}$$

where $L_n^{\alpha-n}(x)$ is a pseudo Laguerre polynomials.
Definition (Discrete orthogonality). Jordan [1] defined orthogonal polynomials with respect to $x = x_0, x_1, \ldots, x_{N-1}$. The polynomials $U(x)$ of degree m are called orthogonal with respect to $x = x_0, x_1, \ldots, x_{N-1}$, if

$$\sum_{i=0}^{N} U_m(x_i)U_\mu(x_i) = 0, \text{ if } m \neq \mu \tag{1.14}$$

If the values of x_i are equidistant, $x = a + h\xi$, and $\xi = 0, 1, 2, \ldots, N-1$, then

$$\sum_{\xi=0}^{N} F_{m-1}(x)U_m(x) = 0, \tag{1.15}$$

where $F_{m-1}(x)$ is an arbitrary polynomial of degree $m-1$. Satyanarayana [11] defined discrete orthogonality and pseduo orthogonality as follows:

Consider a simple set of real polynomials $\phi_n(x)$. If there exists discrete variables $x_i = (x + iw)$, where $i = 0, 1, 2, \ldots, N-1$ and a jump function $J(x_i)$ for each x_i such that each $J(x_i)$ is positive and $\sum_i J(x_i)$ is finite and, if

$$\sum_{i=0}^{N} J(x_i)\phi_n(x_i)\phi_m(x_i) = 0, \text{ if } m \neq n \tag{1.16}$$

we say that the polynomials $\phi_n(x)$ are orthogonal with respect to the jump function $J(x_i)$ for each x_i, because we have taken $J(x_i) > 0$ and $\phi_n(x_i)$ real, it follows that

$$\sum_{i=0}^{N} J(x_i)\{\phi_n(x_i)\}^2 \neq 0 \tag{1.17}$$

Definition (Pseudo-orthogonality). Consider a simple set of real polynomials $\phi_n(x)$. Suppose that there exists discrete variables $x_i = (x + iw)$, with $i = 0, 1, 2, \ldots, N-1$, jump functions $J(x_i; n)J(x_i; m)$ whose values are positive and the sums $\sum_i J(x_i; m)$ and $\sum_i J(x_i; n)$ are finite.

If

$$\sum_{i=0}^{N} J(x_i; n)\phi_n(x_i)\phi_m(x_i) = 0; \quad n > m \tag{1.18}$$

and

$$\sum_{i=0}^{N} J(x_i; m)\phi_n(x_i)\phi_m(x_i) = 0; \quad n < m \tag{1.19}$$

then we say that the polynomials $\phi_n(x)$ are pseudo orthogonal with respect to the jump functions $J(x_i; m)$ and $J(x_i; n)$ for each x_i.

2. Main Results

We define discrete orthogonality for the functions $\{I_n^\alpha(x,w)\}$ and pseudo orthogonality for the functions $\{H_n^\alpha(x,w)\}$ with value $p = q = 1$. Further, we suppose that $a_1 = a$ and $b_1 = a - r$.

Discrete orthogonality for $I_n^\alpha(x,w)$

$$\sum_{x=[\mu w]}^{\infty} I_{n;\lambda;a-r}^{\alpha;\mu;a}(x,w) \cdot I_{m;\lambda;a-r}^{\alpha;\mu;a}(x,w)(x-\mu w)^{[\alpha w]}(1-w)-\frac{x}{w}$$

$$= 0, \text{ if } m \neq n \tag{2.1}$$

$$= \frac{(a)_n(1-w)^{-\lambda}}{(a-r)_n n!}(\lambda_1, -\lambda_2), \tag{2.2}$$

where

$$\lambda_1 = \lim_{x\to\infty} \Delta_{x,w}^{-1}\phi(x+nw,\mu,\lambda,\alpha+n,w),$$
$$\lambda_2 = \lim_{x\to[\mu w]} \Delta_{x,w}^{-1}\phi(x+nw,\mu,\lambda,\alpha+n,w)$$

where

$$\phi(x,\mu,\lambda,\alpha+n,w) = (x-\mu w)^{[(\alpha+n)w]}\cdot(1-w)\frac{x}{w}-\lambda$$
$$\cdot{}_2F_1(-r,-\frac{x}{w}+\lambda;a-r;-\frac{w}{1-w}) \tag{2.3}$$

Pseudo orthogonality for $H_n^\alpha(x,w)$:

$$\sum_{x=[\mu w]}^{\infty} H_{n;\lambda;a-r}^{\alpha;\mu;a}(x,w)\cdot H_{m;\lambda;a-r}^{\alpha;\mu;a}(x,w)\cdot\delta(x,\mu,w)$$

$$= 0, \text{ if } m \neq n \tag{2.4}$$

$$= \frac{(a)_n(1-w)^{-\lambda}}{n!(a-r)_n}(\eta_1-\eta_2), \text{ if } n = m \tag{2.5}$$

where

$$\delta(x,\mu,w) = (x-\mu w)^{[(\alpha+m)w]}\cdot(x-\mu w-\alpha w+nw)^{[2nw]}(1-w)-\frac{x}{w}$$
$$= (-1)^n w^{2n}(1-w)-\frac{x}{w}(x-\mu w)^{[(\alpha+m)w]}$$
$$\cdot(\frac{x}{w}-\mu-\alpha+1)_n(-\frac{x}{w}+\mu+\alpha)_n,$$
$$\eta_1 = \lim_{x\to\infty}\Delta_{x,w}^{-1}\phi(x+nw+w,\mu,\lambda,\alpha,w)$$

and

$$\eta_2 = \lim_{x\to[w\mu]}\Delta_{x,w}^{-1}\phi(x+nw+w,\mu,\lambda,\alpha,w).$$

Proof 2.1. To prove (2.1) consider

$$S = \sum_{x=[\mu w]}^{\infty} I_{n;\lambda;a-r}^{\alpha;\mu;a}(x,w) \cdot I_{m;\lambda;a-r}^{\alpha;\mu;a}(x,w)(x-\mu w)^{[\alpha w]}(1-w) - \frac{x}{w}$$

$$= \Delta_{x,w}^{-1}\left\{ I_{n;\lambda;a-r}^{\alpha;\mu;a}(x,w) I_{m;\lambda;a-r}^{\alpha;\mu;a}(x,w).(x-\mu w)^{[\alpha w]}(1-w) - \frac{x}{w}\right\}\Big|_{x=[\mu w],\infty}$$

using (1.7),(1.8) and (1.4), we have

$$S = \frac{(1+\alpha)_n(1-w)^{-\lambda}}{n!m!}\sum_{p=0}^{\infty}(-1)^p \Delta_{x,w}^{b} F_{1:1;0}^{0:3;2}\left[\begin{matrix} - : -n, a, \frac{x}{w}-\mu+1; \\ a-r : 1+\alpha; \end{matrix}\right.$$

$$\left.\begin{matrix} -r, -\frac{x}{w}+\lambda; \\ \\ ---; \end{matrix} \quad w, -\frac{w}{1-w}\right] \cdot \Delta_{x,w}^{-p-1}\left\{\Delta_{x,w}^{m}\phi(x+pw,\mu,\lambda,\alpha+m,w)\right\}\Big|_{x=[\mu w],\infty}$$

$$= \frac{(1+\alpha)_n(1-w)^{-\lambda}}{n!m!}\sum_{p=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{r}\sum_{i=0}^{p}\sum_{l=0}^{m-p-1}\sum_{h=0}^{l}\binom{p}{i}\binom{l}{n}$$

$$\cdot\frac{(-n)_k(a)_k(r)_j(-1)^p w^k(-w)^j}{(a-r)_{k+j}(1+\alpha)_k(1-w)^j k!j!}\left\{\Delta_{x,w}^{p-i}(\frac{x}{w}-\mu+i+l)_k\right.$$

$$\cdot\Delta_{x,w}^{i}(-\frac{x}{w}+\lambda)_j \cdot \Delta_{x,w}^{m-p-l-1}(x-\mu w+pw+lw)^{[(\alpha+m)w]}$$

$$\left.\cdot\Delta_{x,w}^{l-h}(1-w)\frac{x}{w}-\lambda+h\cdot\Delta_{x,w}^{h}{}_2F_1(-r,-\frac{x}{w}+\lambda;a-r;-\frac{w}{1-w})\right\}\Big|_{x=[\mu w],\infty}$$

In view of (1.2) and (1.5), and by applying $x \to \mu w$ and $x \to \infty$, we get (2.1). Hence the function $\{I_{n;\lambda;a-r}^{\alpha;\mu;a}(x,w)\}$ is discrete orthogonal.

Moreover, if $m = n$, then the series will get the following form.

$$S = \frac{(1+\alpha)_n(1-w)^{-\lambda}}{n!n!}\Delta_{x,w}^{-1}\left\{F_{1:1;0}^{0:3;2}\left[\begin{matrix} --- : -n, a, \frac{x}{w}-\mu+1; \\ a-r : 1+\alpha; \end{matrix}\right.\right.$$

$$\left.\left.\begin{matrix} -r.-\frac{x}{w}+\lambda; \\ \\ ---; \end{matrix} \quad w, -\frac{w}{1-w}\right] \cdot \Delta_{x,w}^{n}\phi(x+pw,\mu,\lambda,\alpha+n,w)\right\}\Big|_{x=[\mu w],\infty}$$

$$= \frac{(1+\alpha)_n(1-w)^{-\lambda}}{n!n!}\sum_{p=0}^{n}(-1)^p \Delta_{x,w}^{p} F_{1:1;0}^{0:3;2}\left[\begin{matrix} --- : -n, a, \frac{x}{w}-\mu+1; \\ a-r : 1+\alpha; \end{matrix}\right.$$

$$\left.\begin{matrix} -r, -\frac{x}{w}+\lambda; \\ \\ ---; \end{matrix} \quad w, -\frac{w}{1-w}\right] \cdot \Delta_{x,w}^{-p-1}(\Delta_{x,w}^{n}\phi(x+pw,\mu,\lambda,\alpha+n,w)\Big|_{x=[\mu w],\infty}$$

$$= \frac{(1+\alpha)_n(-1)^n(1-w)^{-\lambda}}{n!n!}\left\{\Delta_{x,w}^{n} F_{1:1;0}^{0:3;2}\left[\begin{matrix} --- : -n, a, \frac{x}{w}-\mu+1; \\ a-r : 1+\alpha; \end{matrix}\right.\right.$$

$$\left.\left.\begin{matrix} -r, -\frac{x}{w}+\lambda; \\ \\ ---; \end{matrix} \quad w, -\frac{w}{1-w}\right] \cdot \Delta_{x,w}^{-1}\phi(x+nw,\mu,\lambda,\alpha+n,w)\right\}\Big|_{x=[\mu w],\infty}$$

Hence, finally we get (2.2).
Applications: In (2.2), by substituting $r = 0$, $\mu = 1$, $\lambda = 0$ (in view of (1.11)), we get

$$\sum_{x=[\mu w]}^{\infty} \left\{J_n^{\alpha}(x,w)\right\}^2 (1-w)^{\frac{x}{w}} (x-\mu w)^{[\alpha w]}$$
$$= \frac{1}{n!}\Delta_{x,w}^{-1}(x+nw-w)^{[(\alpha+n)w]}(1-w)^{\frac{x}{w}} \Big|_{x=[\mu w],\infty}\,, \tag{2.6}$$

where $J_n^{\alpha}(x,w)$ is the modified Jacobi polynomial (see Lahiri and Satyanaryana [2, p.164(1.5)]. The result (2.6) is believed to be a new result.

Now setting $r = 0$ and letting $w \to 0$, the relation (2.2) is reduced to a well-known result for the orthogonality for the Laguerre polynomials.

Proof 2.4. To prove (2.4) consider the series

$$S_1 = \sum_{x=[\mu w]}^{\infty} H_{n;\lambda;a-r}^{\alpha;\mu;a}(x,w).H_{n;\lambda;a-r}^{\alpha;\mu;\;;a}(x,w).\delta(x,\mu,w)$$

using (1.9) and (1.10), we have

$$S_1 = \frac{(-\alpha)_n(1-w)^{-\lambda}(-1)^n}{n!m!}\Delta_{x,w}^{-1}\left\{F_{1:1;0}^{0:3;2}\left[\begin{matrix} ---:-n,a,\frac{x}{w}-\mu+1; \\ a-r:1+\alpha-n; \end{matrix}\right.\right.$$
$$\left.\left.\begin{matrix} -r,-\frac{x}{w}+\lambda; \\ \\ ---; \end{matrix}\; w,-\frac{w}{1-w}\right] \cdot \Delta_{x,w}^{m}\phi(x,\mu,\lambda,\alpha,w)\right\}\Big|_{x=[\mu w],\infty}$$
$$= \frac{(-\alpha)_n(1-w)^{-\lambda}(-1)^n}{n!m!}\sum_{p=0}^{\infty}(-1)^p\left\{\Delta_{x,w}^{p}F_{1:1;0}^{0:3;2}\left[\begin{matrix} ---:-n,a,\frac{x}{w}-\mu+1; \\ a-r:1+\alpha-n; \end{matrix}\right.\right.$$
$$\left.\left.\begin{matrix} -r,-\frac{x}{w}+\lambda; \\ \\ ---; \end{matrix}\; w,\frac{-w}{1-w}\right] \cdot \Delta_{x,w}^{m-p-1}\phi(x+pw,\mu,\lambda,\alpha,w)\right\}\Big|_{x=[\mu w],\infty}$$

From which the result (2.4) can be deduced directly.

Proof 2.5. Similar to that of (2.2)

Application. By taking $r = 0$ and $w \to 0$, we get

$$\int_0^{\infty}\left\{L_n^{(\alpha-n)}(x)\right\}^2 e^{-x}x^{\alpha-n}dx = \frac{1}{n!}\int_0^{\infty} x^{\alpha}e^{-x}dx$$
$$= \Gamma\frac{(1+\alpha)}{n!} \tag{2.7}$$

where $L_n^{(\alpha-n)}(x)$ is a pseudo Laguerre polynomial.

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