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ON REFINEMENTS OF HADAMARD'S INEQUALITIES

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Abstract. Some refinements of Hadamard's inequalities are established.

1. Introduction

The inequalities

$$f\Big(\frac{a+b}{2}\Big) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

which holds for all convex functions $f : [a,b] \to R$ are known in the literature as Hadamard's inequalities. In [2] and [3], S. S. Dragomir established some refinements of the first inequality of (1.1). In [4], G. S. Yang and M. C. Hong established a refinement of the second inequality of (1.1).

The main purpose of this note is to establish further generalization of the results in [2], [3] and [4].

As in [1] and [2], let E be a nonempty set and let L be a linear class of real-valued functions from E to R having the properties:

$$L_1 : f,g \in L \Rightarrow (af+bg) \in L \text{ for all } a,b \in R;$$
$$L_2 : 1 \in L, \text{ that is, if } f(t) = 1 (t \in E), \text{ then } f \in L.$$

A linear functional $A : L \to R$ is isotonic if

$$A_1 : A(af+bg) = aA(f) + bA(g) \text{ for } f,g \in L \text{ and } a,b \in R;$$
$$A_2 : f \in L, \quad f(t) \ge 0 \text{ on } E \Rightarrow A(f) \ge 0 \quad (A \text{ is isotonic}).$$

We need the following Jensen's inequality (see [1] or [2]).

Jensen's inequality. Let L satisfy the above properties on E, and suppose Φ is a convex function on an interval $I \subseteq R$. If A is any isotonic linear functional with $A(1) = 1$, then, for all $g \in L$ such that $\Phi(g) \in L$, we have $A(g) \in I$ and $\Phi\big(A(g)\big) \le A\big(\Phi(g)\big)$.

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2. Preliminary Lemmas

In order to establish the main theorems, we start with the following lemmas.

Lemma 1. *Let C be a convex subset of a real linear space X, and $f : C \to R$, the real numbers, be a convex function. Let $a_i > 0$ $(i = 1, 2, \dots, n)$ with $\sum_{i=1}^n a_i = 1$ and $a = \min_{1\le i\le n}\{a_i\}$. Given a sequence $x = \{x_i, x_2, \dots, x_n\}$ in C, let $\Phi_x : [0, na] \to R$ be defined by*

$$\Phi_x(t) = \sum_{i=1}^n a_i f\Big(\Big[1 - \frac{g(t)}{na_i}\Big]x_i + \frac{g(t)}{na_i}x_{i+1}\Big),$$

where g is a linear function on $[0, na]$ such that $0 \le g(t) \le na$ and $x_{n+1} = x_1$. Then

(1) *Φ_x is convex on $[0, na]$,*

(2) $$f\Big(\sum_{i=1}^n a_i x_i\Big) \le \Phi_x(t) \le \sum_{i=1}^n a_i f(x_i) \text{ for all } t \in [0, na]. \tag{2.1}$$

Proof. Let $t_1, t_2 \in [0, na]$ and $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$. Since f is convex on C and g is linear in $[0, na]$, we have

$$\begin{aligned}
\Phi_x(\alpha t_1 + \beta t_2) &= \sum_{i=1}^n a_i f\Big(\Big[1 - \frac{g(\alpha t_1 + \beta t_2)}{na_i}\Big]x_i + \frac{g(\alpha t_1 + \beta t_2)}{na_i}x_{i+1}\Big) \\
&= \sum_{i=1}^n a_i f\Big(\alpha\Big[\Big(1 - \frac{g(t_1)}{na_i}\Big)x_i + \frac{g(t_1)}{na_i}x_{i+1}\Big] \\
&\qquad + \beta\Big[\Big(1 - \frac{g(t_2)}{na_i}\Big)x_i + \frac{g(t_2)}{na_i}x_{i+1}\Big]\Big) \\
&\le \alpha \sum_{i=1}^n a_i f\Big(\Big[1 - \frac{g(t_1)}{na_i}\Big]x_i + \frac{g(t_1)}{na_i}x_{i+1}\Big) \\
&\qquad + \beta \sum_{i=1}^n a_i f\Big(\Big[1 - \frac{g(t_2)}{na_i}\Big]x_i + \frac{g(t_2)}{na_i}x_{i+1}\Big) \\
&= \alpha\Phi_x(t_1) + \beta\Phi_x(t_2).
\end{aligned}$$

This completes the proof of (1).

Next, using the convexity of f and note that $x_{n+1} = x_1$, we have

$$\begin{aligned}
\Phi_x(t) &\le \sum_{i=1}^n a_i \Big[\Big(1 - \frac{g(t)}{na_i}\Big)f(x_i) + \frac{g(t)}{na_i}f(x_{i+1})\Big] \\
&= \sum_{i=1}^n a_i f(x_i) + \frac{g(t)}{n}\sum_{i=1}^n [f(x_{i+1}) - f(x_i)] = \sum_{i=1}^n a_i f(x_i)
\end{aligned}$$

and

$$\Phi_x(t) \geq f\left(\sum_{i=1}^{n} a_i\left[\left(1-\frac{g(t)}{na_i}\right)x_i + \frac{g(t)}{na_i}x_{i+1}\right]\right)$$
$$= f\left(\sum_{i=1}^{n} a_i x_i + \frac{g(t)}{n}\sum_{i=1}^{n}[x_{i+1}-x_i]\right) = f\left(\sum_{i=1}^{n} a_i x_i\right),$$

for all $t \in [0, na]$. This proves (2).

Remark 1. Lemma 2.1 in [2] is the special case of our lemma 1 when $n = 2$, $g(t) = t$ and $a_1 = a_2 = \frac{1}{2}$.

In [4], G. S. Yang and M. C. Hong proved:

Lemma 2. *If $f : [a, b] \to R$ is a convex function and $F : [0, 1] \to R$ is defined by*

$$F(t) = \frac{1}{2(b-a)}\int_a^b \left\{f\left(\left[\frac{1+t}{2}\right]a + \left[\frac{1-t}{2}\right]x\right) + f\left(\left[\frac{1+t}{2}\right]b + \left[\frac{1-t}{2}\right]x\right)\right\}dx,$$

then F is convex, increasing on $[0, 1]$ and

$$\frac{1}{b-a}\int_a^b f(x)dx = F(0) \leq F(t) \leq F(1) = \frac{f(a)+f(b)}{2}.$$

They used the differentiability of f on $(0, 1)$ to prove F is increasing on $[0, 1]$. Here, we give a proof without using the differentiability of f on $(0, 1)$ as follows:

Proof. That F is convex on $[0, 1]$ is easy to verify. Now, if $0 \leq t < 1$, then

$$F(t) = \frac{1}{2(b-a)}\int_a^b \left\{f\left(\frac{[1+t]a+[1-t]x}{2}\right) + f\left(\frac{[1+t]b+[1-t]x}{2}\right)\right\}dx$$
$$= \frac{1}{(1-t)(b-a)}\left\{\int_a^{\frac{a+b}{2}-t(\frac{b-a}{2})} f(x)dx + \int_{\frac{a+b}{2}+t(\frac{b-a}{2})}^b f(x)dx\right\}.$$

Since f is convex, we have

$$F'(t) = \frac{1}{(1-t)^2(b-a)}\left\{\int_a^{\frac{a+b}{2}-t(\frac{b-a}{2})} f(x)dx + \int_{\frac{a+b}{2}+t(\frac{b-a}{2})}^b f(x)dx\right\}$$
$$+\frac{1}{(1-t)(b-a)}\left\{f\left(\frac{a+b}{2} - t\left[\frac{b-a}{2}\right]\right)\left[-\frac{b-a}{2}\right]\right.$$
$$\left.-f\left(\frac{a+b}{2} + t\left[\frac{b-a}{2}\right]\right)\left[\frac{b-a}{2}\right]\right\}$$
$$= \frac{1}{(1-t)^2(b-a)}\left\{\int_a^{\frac{a+b}{2}-t(\frac{b-a}{2})} f(x)dx + \int_{\frac{a+b}{2}+t(\frac{b-a}{2})}^b f(x)dx\right\}$$

$$
\begin{aligned}
&-\frac{1}{2(1-t)}\Big\{f\Big(\frac{a+b}{2}-t\Big[\frac{b-a}{2}\Big]\Big)+f\Big(\frac{a+b}{2}+t\Big[\frac{b-a}{2}\Big]\Big)\Big\}\\
\geq\ &\frac{1}{2(1-t)}\Big\{f\Big(\Big[\frac{3+t}{4}\Big]a+\Big[\frac{1-t}{4}\Big]b\Big)+f\Big(\Big[\frac{1-t}{4}\Big]a+\Big[\frac{3+t}{4}\Big]b\Big)\Big\}\\
&-\frac{1}{2(1-t)}\Big\{f\Big(\Big[\frac{1+t}{2}\Big]a+\Big[\frac{1-t}{2}\Big]b\Big)+f\Big(\Big[\frac{1-t}{2}\Big]a+\Big[\frac{1+t}{2}\Big]b\Big)\Big\}\\
=\ &\frac{1}{2(1-t)}\Big\{f\Big(\Big[\frac{1-t}{4}\Big]a+\Big[\frac{3+t}{4}\Big]b\Big)-f\Big(\Big[\frac{1-t}{2}\Big]a+\Big[\frac{1+t}{2}\Big]b\Big)\Big\}\\
&-\frac{1}{2(1-t)}\Big\{f\Big(\Big[\frac{1+t}{2}\Big]a+\Big[\frac{1-t}{2}\Big]b\Big)-f\Big(\Big[\frac{3+t}{4}\Big]a+\Big[\frac{1-t}{4}\Big]b\Big)\Big\}\\
\geq\ &0.
\end{aligned}
$$

This shows that F is increasing on $[0,1]$. Hence

$$\frac{1}{b-a}\int_a^b f(x)dx=F(0)\leq F(t)\leq F(1)=\frac{f(a)+f(b)}{2}.$$

This completes the proof.

3. Main Results

Now, we give our main results as the following theorems.

Theorem 1. *Under the conditions of Lemma 1, let L, A satisfy the conditions L_1, L_2, A_1 and A_2, and let $h: E\to[0,na]$ be a function such that $h\in L$ and*

$$f\Big(\Big[1-\frac{g(h)}{na_i}\Big]x_i+\frac{g(h)}{na_i}x_{i+1}\Big)\in L \qquad \text{for } i=1,2,\ldots,n.$$

If $A(1)=1$, then

$$f\Big(\sum_{i=1}^n a_ix_i\Big)\leq\Phi_x\big(A(h)\big)\leq A\big(\Phi_x(h)\big)\leq\sum_{i=1}^n a_if(x_i). \tag{3.1}$$

Proof. Using Jensen's inequality, we have

$$\Phi_x\big(A(h)\big)\leq A\big(\Phi_x(h)\big).$$

This is the second inequality in (3.1).

Since f is convex on C and A is an istonic linear functional on L, we have

$$
\begin{aligned}
\Phi_x\big(A(h)\big)&=\sum_{i=1}^n a_if\Big(\Big[1-\frac{g(A(h))}{na_i}\Big]x_i+\frac{g(A(h))}{na_i}x_{i+1}\Big)\\
&\geq f\left(\sum_{i=1}^n a_i\Big[\Big(1-\frac{g(A(h))}{na_i}\Big)x_i+\frac{g(A(h))}{na_i}x_{i+1}\Big]\right)=f\left(\sum_{i=1}^n a_ix_i\right).
\end{aligned}
$$

This is the first inequality of (3.1).

Finally,

$$\Phi_x(h) = \sum_{i=1}^{n} a_i f\Big(\Big[1 - \frac{g(h)}{na_i}\Big]x_i + \frac{g(h)}{na_i}x_{i+1}\Big)$$
$$\le \sum_{i=1}^{n} a_i\Big[\Big(1 - \frac{g(h)}{na_i}\Big)f(x_i) + \frac{g(h)}{na_i}f(x_{i+1})\Big] = \sum_{i=1}^{n} a_i f(x_i).$$

Using A_1, A_2 and $A(1) = 1$, we have $A\big(\Phi_x(h)\big) \le A\left(\sum_{i=1}^{n} a_i f(x_i)\right) = \sum_{i=1}^{n} a_i f(x_i)$. This proves the last inequality of (3.1).

Remark 2. We note that Theorem 2.3 in [2] is the special case of Theorem 1 as $n = 2$, $a_1 = a_2 = \frac{1}{2}$, $g(t) = t$ and $h(t) = t$.

Theorem 2. *Under the conditions of Lemma 1, if $x = \{x_1, x_2, \ldots, x_n\}$ is a sequence in C such that $x_i \ne x_{i+1}$, $i = 1, 2, \ldots, n$, and $x_{n+1} = x_1$, then*

$$f\Big(\sum_{i=1}^{n} a_i x_i\Big) \le \sum_{i=1}^{n} a_i f\Big(\Big[1 - \frac{a}{2a_i}\Big]x_i + \frac{a}{2a_i}x_{i+1}\Big)$$
$$\le \sum_{i=1}^{n} \frac{a_i^2}{a(x_{i+1} - x_i)} \int_{x_i}^{x_i + \frac{a}{a_i}(x_{i+1} - x_i)} f(t)dt$$
$$\le \sum_{i=1}^{n} a_i f(x_i). \tag{3.2}$$

Proof. Let $A = \frac{1}{na}\int_0^{na}$, $E = [0, na]$, $g(t) = t$ and $h(t) = t$. Then

$$A\big(\Phi_x(h)\big) = \frac{1}{na}\int_0^{na} \sum_{i=1}^{n} a_i f\Big(\Big[1 - \frac{t}{na_i}\Big]x_i + \frac{t}{na_i}x_{i+1}\Big)dt$$
$$= \sum_{i=1}^{n} \frac{a_i^2}{a(x_{i+1} - x_i)} \int_{x_i}^{x_i + \frac{a}{a_i}(x_{i+1} - x_i)} f(t)dt,$$

and

$$\Phi_x\big(A(h)\big) = \sum_{i=1}^{n} a_i f\left(\left[1 - \frac{\frac{1}{na}\int_0^{na} tdt}{na_i}\right]x_i + \frac{\frac{1}{na}\int_0^{na} tdt}{na_i}x_{i+1}\right)$$
$$= \sum_{i=1}^{n} a_i f\Big(\Big[1 - \frac{a}{2a_i}\Big]x_i + \frac{a}{2a_i}x_{i+1}\Big).$$

Using (3.1), we obtain

$$f\Big(\sum_{i=1}^{n} a_i x_i\Big) \le \sum_{i=1}^{n} a_i f\Big(\Big[1 - \frac{a}{2a_i}\Big]x_i + \frac{a}{2a_i}x_{i+1}\Big)$$

$$\leq \sum_{i=1}^{n} \frac{a_i^2}{a(x_{i+1}-x_i)} \int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)} f(t)dt$$
$$\leq \sum_{i=1}^{n} a_i f(x_i).$$

Remark 3. We note that the Hadamard's inequalities (1.1) is the special case of Theorem 2 when $n=2$, $x_1=a$, $x_2=b$, and $a_1=a_2=\frac{1}{2}$.

Tehorem 3. *Under the conditions of Theorem 2, let $H:[0,1]\to R$ be a function defined by*

$$H(t) = \sum_{i=1}^{n} \frac{a_i^2}{a(x_{i+1}-x_i)} \int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)} f\left(tx+(1-t)\sum_{j=1}^{n} a_j x_j\right) dx.$$

Then (1) *H is convex on* $[0,1]$,

$$(2)\ f\left(\sum_{i=1}^{n} a_i x_i\right) = H(0) = \min_{t\in[0,1]} H(t) \leq H(t)$$
$$\leq \max_{t\in[0,1]} H(t) = H(1) = \sum_{i=1}^{n} \frac{a_i^2}{a(x_{i+1}-x_i)} \int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)} f(x)dx$$
$$\leq \sum_{i=1}^{n} a_i f(x_i), \tag{3.3}$$

for all $t\in[0,1]$,
(3) *H is increasing on* $[0,1]$.

Proof. Let $t_1, t_2 \in [0,1]$ and $\alpha, \beta \geq 0$ with $\alpha+\beta=1$. Since f is convex on C, we have

$$H(\alpha t_1+\beta t_2)$$
$$= \sum_{i=1}^{n} \frac{a_i^2}{a(x_{i+1}-x_i)} \int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)} f\Big((\alpha t_1+\beta t_2)x+(1-\alpha t_1-\beta t_2)\sum_{j=1}^{n} a_j x_j\Big)dx$$
$$= \sum_{i=1}^{n} \frac{a_i^2}{a(x_{i+1}-x_i)}$$
$$\times \int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)} f\Big(\alpha\Big[t_1 x+(1-t_1)\sum_{j=1}^{n} a_j x_j\Big]+\beta\Big[t_2 x+(1-t_2)\sum_{j=1}^{n} a_j x_j\Big]\Big)dx$$
$$\leq \alpha \sum_{i=1}^{n} \frac{a_i^2}{a(x_{i+1}-x_i)} \int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)} f\Big(t_1 x+(1-t_1)\sum_{j=1}^{n} a_j x_j\Big)dx$$

$$+\beta\sum_{i=1}^{n}\frac{a_i^2}{a(x_{i+1}-x_i)}\int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)} f\Big(t_2x+(1-t_2)\sum_{j=1}^{n}a_jx_j\Big)dx$$
$$=\alpha H(t_1)+\beta H(t_2).$$

This completes the proof of (1).

Now, observe that $H(0)=f\Big(\sum_{i=1}^{n}a_ix_i\Big)$ and $H(1)=\sum_{i=1}^{n}\frac{a_i^2}{a(x_{i+1}-x_i)}$ $\int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)} f(x)dx$. Using the convexity of f and the inequality (3.2), we have

$$\begin{aligned}H(t)&=\sum_{i=1}^{n}\frac{a_i^2}{a(x_{i+1}-x_i)}\int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)} f\Big(tx+(1-t)\sum_{j=1}^{n}a_jx_j\Big)dx\\
&\le t\sum_{i=1}^{n}\frac{a_i^2}{a(x_{i+1}-x_i)}\int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)} f(x)dx\\
&\qquad+(1-t)\sum_{i=1}^{n}\frac{a_i^2}{a(x_{i+1}-x_i)}\int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)} f\Big(\sum_{j=1}^{n}a_jx_j\Big)dx\\
&=t\sum_{i=1}^{n}\frac{a_i^2}{a(x_{i+1}-x_i)}\int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)} f(x)dx+(1-t)f\Big(\sum_{j=1}^{n}a_jx_j\Big)\\
&\le\sum_{i=1}^{n}\frac{a_i^2}{a(x_{i+1}-x_i)}\int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)} f(x)dx\le\sum_{i=1}^{n}a_if(x_i),\end{aligned}$$

for all $t\in[0,1]$.

On the other hand, let $y_i=tx_i+(1-t)\sum_{j=1}^{n}a_jx_j$, $1\le i\le n$, and $y_{n+1}=y_1$, then

$$\begin{aligned}H(t)&=\sum_{i=1}^{n}\frac{a_i^2}{a(y_{i+1}-y_i)}\int_{y_i}^{y_i+\frac{a}{a_i}(y_{i+1}-y_i)} f(y)dy\\
&\ge f\Big(\sum_{j=1}^{n}a_iy_i\Big)=f\Big(\sum_{i=1}^{n}a_ix_i\Big),\end{aligned}$$

for all $t\in[0,1]$.

This completes the proof of (2).

Finally, let $0<t<u\le 1$. Since H is convex on $[0,1]$ and $H(t)\ge H(0)$, we have

$$\frac{H(u)-H(t)}{u-t}\ge\frac{H(t)-H(0)}{t}\ge 0,$$

that is $H(t)\le H(u)$.

This completes the proof of (3).

Remark 4. We note that Theorem 1 in [3] is the special case of Theorem 3 when $n=2$, $a_1=a_2=\frac{1}{2}$.

Theorem 4. *Under the conditions of Theorem 2, let $K : [0,1] \to R$ be a function defined by*

$$K(t) = \sum_{i=1}^{n} \frac{a_i^2}{2a(x_{i+1}-x_i)} \int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)} \Big\{ f\Big(\Big[\frac{1-t}{2}\Big]x + \Big[\frac{1+t}{2}\Big]x_i\Big)$$
$$+ f\Big(\Big[\frac{1-t}{2}\Big]x + \Big[\frac{1+t}{2}\Big]\Big[x_i + \frac{a}{a_i}(x_{i+1}-x_i)\Big]\Big) \Big\} dx$$

Then K is convex, increasing on $[0,1]$ and

$$\sum_{i=1}^{n} \frac{a_i^2}{a(x_{i+1}-x_i)} \int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)} f(x)dx = K(0) \le K(t) \le K(1) \le \sum_{i=1}^{n} a_i f(x_i)$$

for all $t \in [0,1]$.

Proof. Using Lemma 2, it is easy to see that K is convex and increasing on [0,1]. Now,

$$K(0) = \sum_{i=1}^{n} \frac{a_i^2}{a(x_{i+1}-x_i)} \int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)} f(x)dx$$
$$K(1) = \sum_{i=1}^{n} \frac{a_i^2}{a(x_{i+1}-x_i)}$$
$$\int_{x_i}^{x_i+\frac{a}{a_i}(x_{i+1}-x_i)} \frac{1}{2}\Big[f(x_i) + f\Big(x_i + \frac{a}{a_i}(x_{i+1}-x_i)\Big)\Big]dx$$
$$= \sum_{i=1}^{n} \frac{a_i\Big[f(x_i) + f\Big(x_i + \frac{a}{a_i}(x_{i+1}-x_i)\Big)\Big]}{2}$$
$$= \frac{1}{2}\sum_{i=1}^{n} a_i f(x_i) + \frac{1}{2}\sum_{i=1}^{n} a_i f\Big(x_i + \frac{a}{a_i}(x_{i+1}-x_i)\Big)$$
$$\le \sum_{i=1}^{n} a_i f(x_i).$$

This completes the proof.

Remark 5. Let $a_i = \frac{1}{n} (i = 1, 2, \ldots, n)$ and $x_{n+1} = x_1 < x_2 < \cdots < x_n$. Then, from Theorem 3 and Theorem 4, we have

$$H(t) = \sum_{i=1}^{n} \frac{1}{n(x_{i+1}-x_i)} \int_{x_i}^{x_{i+1}} f\Big(tx + \frac{1-t}{n}\sum_{j=1}^{n} x_j\Big)dx,$$

and

$$K(t)=\sum_{i=1}^{n}\frac{1}{2n(x_{i+1}-x_i)}\int_{x_i}^{x_{i+1}}\Big\{f\Big(\Big[\frac{1-t}{2}\Big]x+\Big[\frac{1+t}{2}\Big]x_i\Big)$$
$$+f\Big(\Big[\frac{1-t}{2}\Big]x+\Big[\frac{1+t}{2}\Big]x_{i+1}\Big)\Big\}dx$$

such that H and K are convex increasing on $[0,1]$ and

$$\begin{aligned}f\Big(\frac{1}{n}\sum_{i=1}^{n}x_i\Big)&=H(0)\le H(t)\\&\le H(1)=\sum_{i=1}^{n}\frac{1}{n(x_{i+1}-x_i)}\int_{x_i}^{x_{i+1}}f(x)dx=K(0)\\&\le K(t)\le K(1)=\frac{1}{n}\sum_{i=1}^{n}f(x_i).\end{aligned}$$

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