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ON HUA'S INEQUALITY FOR COMPLEX NUMBERS

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Abstract. A generalization is given of Hua's inequality for complex numbers that involves convex functions. An improvement is derived for Hua's inequality and for two recent extensions of it to the complex domain.

1. Introduction

Suppose δ, α are positive numbers, $x_i \in \mathbb{R}(i=1,\dots,n)$ and $K_n = \alpha(n+\alpha)^{-1}$. Hua [4] has shown that

$$\Big(\delta - \sum_{i=1}^{n} x_i\Big)^2 + \alpha \sum_{i=1}^{n} x_i^2 \geq K_n \delta^2,$$

with equality if and only if $x_i = h_n\delta$, where $h_n = (n+\alpha)^{-1}$. This inequality is important in number theory.

Wang [7] has given the following generalization.
Let α, δ be as above and $p>1$. Then

$$\Big(\delta - \sum_{i=1}^{n} x_i\Big)^p + \alpha^{p-1} \sum_{i=1}^{n} x_i^p \geq K_n^{p-1} \delta^p \tag{1.1}$$

holds for all nonnegative $x_i \in \mathbb{R}$ $(i=1,\dots,n)$ with $\sum_{i=1}^n x_i < \delta$. The inequality is reversed for $0<p<1$. In either case, equality holds if and only if $x_i = h_n\delta$ $(i=1,\dots,n)$.

Moreover, Pearce and Pečarić [6] have proved that if $\sum_{i=1}^n x_i \leq \delta$ and $x_i > 0$ $(i=1,\dots,n)$, then (1.1) is also valid for $p<0$. This follows as a special case of a more general result.

Let f be a real-valued, convex function on an interval $F \subseteq \mathbb{R}$ and let $\alpha, \delta, x_1, \dots, x_n$ be real numbers such that $\alpha > 0$ and $\delta - x_1 - \cdots - x_n, \alpha x_1, \dots, \alpha x_n \in F$. Then

$$f\Big(\delta - \sum_{i=1}^{n} x_i\Big) + \sum_{i=1}^{n} \alpha^{-1} f(\alpha x_i) \geq \frac{\alpha+n}{\alpha} f\Big(\frac{\alpha\delta}{\alpha+n}\Big). \tag{1.2}$$

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If f is strictly convex, equality applies if $x_1 = \cdots = x_n = \cdots = h_n\delta$, where h_n is defined as above.

Hua's inequality has been transported to the complex domain by Dragomir [2].

Theorem A. *Suppose $\alpha > 0$ and $\delta, z_1, \ldots, z_n \in C$. Then*

$$\left|\delta - \sum_{i=1}^{n} z_i\right|^2 + \alpha \sum_{i=1}^{n} |z_i|^2 \geq \frac{\alpha|\delta|^2}{n+\alpha}. \tag{1.3}$$

Equality applies if and only if $z_i = \delta/(n+\alpha)$ $(i = 1, \ldots, n)$.

Dragomir also supplied an interesting extension.

Theorem B. *Suppose $\alpha > 0$ and $\delta, z_i, w_i \in C$ $(i = 1, \ldots, n)$. Then*

$$\left|\delta - \sum_{i=1}^{n} z_i w_i\right|^2 + \alpha \sum_{i=1}^{n} |z_i|^2 \geq \frac{\alpha|\delta|^2}{\alpha + \sum_{i=1}^{n} |w_i|^2}.$$

Equality applies if and only if

$$z_i = \frac{\delta \overline{w_i}}{\alpha + \sum_{i=1}^{n} |w_i|^2}$$

for all $i \in \{1, \ldots, n\}$.

Recently Yang and Han [8] melded Theorem A with Wang's result.

Theorem C. *Suppose $\alpha > 0$ with $\delta, z_1, \ldots, z_n \in C$ and $p > 1$. Then*

$$\left|\delta - \sum_{i=1}^{n} z_i\right|^p + \alpha^{p-1} \sum_{i=1}^{n} |z_i|^p \geq \left(\frac{\alpha}{n+\alpha}\right)^{p-1} |\delta|^p,$$

with equality if and only if $z_i = \delta/(n+\alpha)$ $(i = 1, \ldots, n)$.

In this paper we present a general result which subsumes Theorems A, B and C as special cases. As particular instances we derive improvements of Theorems A and B.

2. Results

Theorem 2.1. *Suppose $\alpha > 0$ and $\delta, z_i, w_i \in C$ with $w_i \neq 0$ $(i = 1, \ldots, n)$. If f is a convex nondecreasing function on $[0, \infty)$, then*

$$f\left(\left|\delta - \sum_{i=1}^{n} z_i w_i\right|\right) + \alpha^{-1} \sum_{i=1}^{n} |w_i| f(\alpha|z_i|) \geq \frac{\alpha + \sum_{i=1}^{n} |w_i|}{\alpha} f\left(\frac{\alpha|\delta|}{\alpha + \sum_{i=1}^{n} |w_i|}\right). \tag{2.1}$$

If f is strictly convex, equality holds if and only if

$$z_i = \frac{\delta \overline{w}_i}{(\alpha + \sum_{i=1}^n |w_i|)|w_i|} \quad (i = 1, \dots, n). \tag{2.2}$$

Proof. By the triangle inequality, we have

$$\left|\delta - \sum_{i=1}^n z_i w_i\right| \geq \left||\delta| - \left|\sum_{i=1}^n w_i z_i\right|\right|. \tag{2.3}$$

Hence since f is nondecreasing on $[0, \infty)$,

$$f\left(\left|\delta - \sum_{i=1}^n z_i w_i\right|\right) \geq f\left(\left||\delta| - \left|\sum_{i=1}^n w_i z_i\right|\right|\right). \tag{2.4}$$

Also Jensen's inequality for convex functions gives

$$\begin{aligned} \alpha^{-1} \sum_{i=1}^n |w_i| f(\alpha|z_i|) &= \frac{\sum_{i=1}^n |w_i|}{\alpha} \cdot \frac{\sum_{i=1}^n |w_i| f(\alpha|z_i|)}{\sum_{i=1}^n |w_i|} \\ &\geq \frac{\sum_{i=1}^n |w_i|}{\alpha} f\left(\frac{\alpha}{\sum_{i=1}^n |w_i|} \sum_{i=1}^n |w_i||z_i|\right) \\ &\geq \frac{\sum_{i=1}^n |w_i|}{\alpha} f\left(\frac{\alpha}{\sum_{i=1}^n |w_i|} \left|\sum_{i=1}^n w_i z_i\right|\right). \end{aligned} \tag{2.5}$$

The last inequality follows from the triangle inequality, since f is nondecreasing on $[0, \infty)$.

Note that $f(|x|)$ is also a convex function.

So by (2.4) and (2.5) we have

$$\begin{aligned} & f\left(\left|\delta - \sum_{i=1}^n z_i w_i\right|\right) + \alpha^{-1} \sum_{i=1}^n |w_i| f(\alpha|z_i|) \\ &\geq f\left(\left||\delta| - \left|\sum_{i=1}^n z_i w_i\right|\right|\right) + \frac{\sum_{i=1}^n |w_i|}{\alpha} f\left(\frac{\alpha}{\sum_{i=1}^n |w_i|} \left|\sum_{i=1}^n w_i z_i\right|\right) \\ &\geq \frac{\alpha + \sum_{i=1}^n |w_i|}{\alpha} f\left(\left|\frac{\alpha|\delta|}{\alpha + \sum_{i=1}^n |w_i|}\right|\right) \\ &= \frac{\alpha + \sum_{i=1}^n |w_i|}{\alpha} f\left(\frac{\alpha|\delta|}{\alpha + \sum_{i=1}^n |w_i|}\right), \end{aligned} \tag{2.6}$$

where we have used (1.2) for $n = 1$ and $x_1 = |\sum_{i=1}^n z_i w_i|$ with the replacements $\delta \to |\delta|$, $\alpha \to \alpha / \sum_{i=1}^n |w_i|$ and $f(x) \to f(|x|)$.

We now address the condition for equality in the event that f is strictly conves. Since it is nondecreasing it must be strictly increasing on $[0, \infty)$. The first inequality in (2.5) holds with equality if and only if

$$|z_1| = \cdots = |z_n|, \tag{2.7}$$

while in the second inequality in (2.6), equality applies if

$$\left|\sum_{i=1}^{n} z_i w_i\right| = \frac{|\delta| \sum_{i=1}^{n} |w_i|}{\alpha + \sum_{i=1}^{n} |w_i|}. \tag{2.8}$$

The second inequality in (2.5) becomes an equality if

$$\left|\sum_{i=1}^{n} w_i z_i\right| = \sum_{i=1}^{n} |w_i||z_i|, \tag{2.9}$$

that is, by (2.7),

$$\left|\sum_{i=1}^{n} w_i z_i\right| = |z_1| \sum_{i=1}^{n} |w_i|. \tag{2.10}$$

Conditions (2.8) and (2.10) yield

$$|z_1| = \cdots = |z_n| = \frac{|\delta|}{\alpha + \sum_{i=1}^{n} |w_i|}. \tag{2.11}$$

If $\delta = 0$, then $z_i = 0$ for each i, and so (2.2) must hold. So suppose $\delta \neq 0$. Now (2.9) holds with equality if and only if

$$z_i w_i = \lambda_i r \qquad (i = 1, \ldots, n) \tag{2.12}$$

for some complex number r with

$$\lambda_i \geq 0 \qquad (i = 1, \ldots, n).$$

In fact (2.11) and the assumption that $w_i \neq 0$ requires that each λ_i be strictly positive. Equality obtains in (2.3) and so also (2.4) and the first part of (2.6) if and only if $\sum_{i=1}^{n} z_i w_i$ is a nonnegative multiple of δ, so that without loss of generality we may take $r = \delta$ in (2.12) to give

$$z_i w_i = \lambda_i \delta \qquad (i = 1, \ldots, n).$$

By (2.11)

$$\lambda_i \left|\frac{\delta}{w_i}\right| = \frac{|\delta|}{\alpha + \sum_{i=1}^{n} |w_i|}$$

and so

$$\lambda_i = \frac{|w_i|}{\alpha + \sum_{i=1}^{n} |w_i|} \qquad (i = 1, \ldots, n).$$

Thus it is necessary that

$$w_i z_i = \frac{|w_i| \delta}{\alpha + \sum_{i=1}^{n} |w_i|} \qquad (i = 1, \ldots, n),$$

that is, that (2.2) holds.

It is trivial that this condition is also sufficient.

Remark. For $f(x) = x^2$ and w_i and z_i replaced by w_i^2 and z_i/w_i respectively, we get Theorem B from Theorem 2.1.

Corollary 2.1. *Suppose $\alpha > 0$ and $\delta, z_i, w_i \in C$ $(i = 1, \dots, n)$. If $p > 1$ and $1/p + 1/q = 1$, then*

$$\left|\delta - \sum_{i=1}^{n} z_i w_i\right|^p + \alpha^{p-1} \sum_{i=1}^{n} |z_i|^p \geq \left(\frac{\alpha}{\alpha + \sum_{i=1}^{n} |w_i|^q}\right)^{p-1} |\delta|^p. \tag{2.13}$$

Equality obtains if and only if $z_i = \frac{\delta \overline{w}_i |w_i|^{q-2}}{\alpha + \sum_{i=1}^{n} |w_i|^q}$.

Proof. Set $f(x) = x^p$ in Theorem 2.1 and replace w_i by w_i^q and z_i by $z_i w_i^{-q/p}$. Then (2.1) becomes (2.13) and the condition for equality becomes

$$z_i w_i^{-q/p} = \frac{\delta \overline{w}_i^q}{|w_i|^q (\alpha + \sum_{k=1}^{n} |w_k|^q)},$$

that is,

$$\begin{aligned} z_i &= \frac{\delta \overline{w}_i^q w_i^{q/p}}{|w_i|^q (\alpha + \sum_{k=1}^{n} |w_k|^q)} \\ &= \frac{\delta |w_i|^{2q} w_i^{-1}}{|w_i|^q (\alpha + \sum_{k=1}^{n} |w_k|^q)} \\ &= \frac{\delta |w_i|^q w_i^{-1}}{\alpha + \sum_{k=1}^{n} |w_k|^q}. \end{aligned}$$

Remark. *For $w_i = 1$ $(i = 1, \dots, n)$, Corollary 2.1 gives Theorem C.*

Theorem B possesses an improvement, Theorem 2.2 below. To establish this, we shall make use of an improvement of Cauchy's inequality is due to de Bruijn [1] (see also [5, pp. 89-90]).

Proposition 2.1. *If $a_1, \dots, a_n$ are real and $z_1, \dots, z_n$ complex numbers, then*

$$\left|\sum_{k=1}^{n} a_k z_k\right|^2 \leq \frac{1}{2}\left(\sum_{k=1}^{n} |z_k|^2 + \left|\sum_{k=1}^{n} z_k^2\right|\right)\left(\sum_{k=1}^{n} a_k^2\right).$$

Equality holds if and only if $a_k = Re(\lambda z_k)$ for $k = 1, \dots, n$, where λ is a complex number and $\sum_{k=1}^{n} \lambda^2 z_k^2$ is real and nonnegative.

Theorem 2.2. *Suppose $\alpha > 0$ and $a_i \in \mathbb{R}$ with $\delta, z_i \in C$ $(i = 1, \dots, n)$. Then*

$$\left|\delta - \sum_{i=1}^{n} a_i z_i\right|^2 + \frac{\alpha}{2}\left(\sum_{i=1}^{n} |z_i|^2 + \left|\sum_{i=1}^{n} z_i^2\right|\right) \geq \frac{\alpha|\delta|^2}{\alpha + \sum_{i=1}^{n} a_i^2}.$$

Equality holds if and only if $a_k = Re(\lambda z_k)$ *for* $k = 1, \ldots, n$, *where* λ *is a complex number,* $\sum_{k=1}^n \lambda^2 z_k^2$ *is real and nonnegative and*

$$\sum_{i=1}^n a_i z_i = \frac{\sum_{i=1}^n a_i^2 \delta}{\alpha + \sum_{i=1}^n a_{i2}}.$$

Proof. If $\sum_{i=1}^n a_i^2 = 0$, then $a_i = 0$ for all $i \in \{1, \ldots, n\}$ and the inequality is obvious, so suppose that $\sum_{i=1}^n a_i^2 > 0$. By Proposition 2.1 we have

$$\begin{aligned} \left|\delta - \sum_{k=1}^n a_k z_k\right|^2 &+ \frac{\alpha}{2}\left(\sum_{k=1}^n |z_k|^2 + \left|\sum_{i=1}^n z_i^2\right|\right) \\ &\geq \left|\delta - \sum_{k=1}^n a_k z_k\right|^2 + \frac{\alpha}{\sum_{k=1}^n a_k^2}\left|\sum_{k=1}^n a_k z_k\right|^2 \\ &\geq \frac{\alpha|\delta|^2}{\alpha + \sum_{k=1}^n a_k^2}. \end{aligned}$$

In the last step we have used (1.3) for $n = 1$ and with z_1, α replaced respectively by $\sum_{k=1}^n \alpha_n z_n$ and $\alpha / \sum_{k=1}^n \alpha_k^2$. The condition for equality is inherited from the corresponding conditions in Proposition 2.1 and Theorem A.

Remark. *For* $a_1 = \cdots = a_n = 1$, *we have*

$$\left|\delta - \sum_{k=1}^n z_k\right|^2 + \frac{\alpha}{2}\left(\sum_{k=1}^n |z_k|^2 + \left|\sum_{k=1}^n z_k^2\right|\right) \geq \frac{\alpha|\delta|^2}{\alpha + n},$$

which is an improvement of (1.3).

Remark. Drnovšek [3] has given an operator generalization of the Lo-Keng Hua inequality.

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