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SOME CLASSES OF ANALYTIC FUNCTIONS RELATED WITH BAZILEVIČ FUNCTIONS

KHALIDA INAYAT NOOR

Abstract. A certain class $M_k(\alpha,\beta)$ of analytic functions is introduced and it is shown that $M_2(\alpha,\beta)$ is contained in the class of Bazilevič functions. Some other properties of $M_k(\alpha,\beta)$ are also derived.

1. Introduction

Let A denote the class of functions $f: f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in the unit disc $E = \{z : |z| < 1\}$. By S, K, S^* and C, we denote the subclasses of A which are respectively univalent, close-to-convex, starlike (with respect to the origin) and convex in E.

Let P_k be the class of analytic function p defined in E and with representation

$$p(z) = \frac{1}{2}\int_{-\pi}^{\pi} \frac{1+ze^{-it}}{1-ze^{-it}} d\mu(t), \tag{1.1}$$

where $\mu(t)$ is a function with bounded variation on $[-\pi,\pi]$ and it satisfies the conditions

$$\int_{-\pi}^{\pi} d\mu(t) = 2, \qquad \int_{-\pi}^{\pi} |d\mu(t)| \leq k. \tag{1.2}$$

We note that $k \geq 2$ and $P_2 = P$ is the class of analytic functions with positive real part in E with $p(0) = 1$. The class P_k was introduced in [3]. From the integral representation (1.1) it is immediately clear that $p \in P_k$ if and only if there are analytic function $p_1, p_2 \in P$ such that

$$p(z) = \Big(\frac{k}{4} + \frac{1}{2}\Big)p_1(z) - \Big(\frac{k}{4} - \frac{1}{2}\Big)p_2(z). \tag{1.3}$$

We defined the Hadamard product or Convolution of two analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^{n+1}$ and $g(z) = \sum_{n=0}^{\infty} b_n z^{n+1}$ as

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^{n+1}, \quad z \in E.$$

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We now define the following.

Definition 1.1. A function $f \in A$ is said to be belong to the class $M_k(\alpha, \beta)$ if and only if it satisties the property

$$J(\alpha,\beta;f(z),g(z)) = \left\{\frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)-\alpha(1-\beta)\frac{zf'(z)}{f(z)}-\alpha\beta\frac{zg'(z)}{g(z)}\right\} \in P_k,$$

for some real $\alpha, \beta(\beta \geq 0)$, $g \in A$ and $z \in E$.

Special Cases

(i) For $k = 2$, $\alpha = 0$ and $g \in S^*$, we obtain the sell-known class $B(\beta) \subset S$ of Bazilevič functions of type β.

(ii) When $\beta = 0$, the class $M_k(\alpha, 0)$ consists of functions with bounded Mocanu variation, see [1] and $M_2(\alpha, 0)$ is the class of α-starlike functions which are known to be starlike univalent in E.

(iii) $M_k(1, 0) = V_k$ is the class of functions with bounded boundary rotation and $M_2(1, 0) = C$.

(iv) $M_k(0, 0)$ is the class R_k of functions with bounded radius rotation and $M_2(0, 0) = S^*$.

(v) For $\alpha = 0$, $\beta = 1$ and $g \in R_k$, we have $M_2(0, 1) = T_k$ which has been introduced and studied in [2]. Also for $g \in S^*$, we note that $M_2(0, 1) = K$.

2. Preliminary Results

Lemma 2.1. *Let p be analytic in E and $p(0) = 1$. Then, for $\alpha \geq 0$, $z \in E$, $(p + \alpha\frac{zp'}{p}) \in P_k$ implies $p \in P_k$ in E.*

The proof follows directly from the result, proved in [1], that functions with bounded Mocanu variation are in R_k.

From the Herglotz representation (1.1) for $k = 2$, we have the following result.

Lemma 2.2. *If p is analytic in E, $p(0) = 1$ and $Re p(z) > \frac{1}{2}$, $z \in E$, then for any function F, analytic in E, the function $p * F$ takes values in the convex hull of the image of E under F.*

3. Main Results

Theorem 3.1. *For $\alpha \geq 0$, $M_k(\alpha, \beta) \subset M_k(0, \beta)$.*

Proof. Let $f \in M_k(\alpha, \beta)$. We define

$$\frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)} = H(z). \tag{3.1}$$

We see that $H(0) = 1$ and H is analytic in E. Logarithmic differentiation of (3.1) gives us

$$\alpha\left(1+\frac{zf''(z)}{f'(z)}\right) - \alpha(1-\beta)\frac{zf'(z)}{f(z)} - \alpha\beta\frac{zg'(z)}{g(z)} = \alpha z\frac{H'(z)}{H(z)},$$

and so

$$J(\alpha,\beta;f,g) = H(z) + \alpha\frac{zH'(z)}{H(z)}.$$

Since $f \in M_k(\alpha,\beta)$, it follows that $(H + \alpha\frac{zH'}{H}) \in P_k$ and using Lemma 2.1, we conclude that $H \in P_k$. Consequently $f \in M_k(0,\beta)$.

Corollary 3.1. *Let $g \in S^*$ and $k = 2$. Then, for $\alpha \geq 0$,*

$$M_2(\alpha,\beta) \subset S$$

since in this case $\frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)} \in P$, $g \in S^*$ implies f is Bazilevič and hence univalent.

Corollary 3.2. *For $\alpha \geq 0$, $M_k(\alpha,0) \subset M_k(0,0)$. That is a function with bounded Mocanu variation is of bounded radius rotation. With $k = 2$ we also deduce that α-starlike functions are starlike.*

Corollary 3.3. *Let $g \in S^*$ and $k = 2$, $\beta = 1$. Then $M_2(\alpha,1) \subset M_2(0,1) = K \subset S$. That is $f \in M_2(\alpha,1)$ is a close-to-convex univalent function.*

In the opposite direction we prove the following.

Theorem 3.2. *Let $f \in M_k(0,\beta)$. Then $f \in M_k(\alpha,\beta)$ for $|z| < r_\alpha$ where*

$$r_\alpha = \frac{1}{2\alpha + \sqrt{4\alpha^2 - 2\alpha + 1}}. \tag{3.2}$$

Proof. With $\frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)} = H(z) \in P_k$, we have

$$J(\alpha,\beta;f,g) = H(z) + \alpha\frac{zH'(z)}{H(z)}. \tag{3.3}$$

Since $H \in P_k$, we use (1.3) to write

$$H(z) = \left(\frac{k}{4}+\frac{1}{2}\right)h_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)h_2(z), \quad h_1, h_2 \in P.$$

$$\text{Let } \phi_\alpha(z) = (1-\alpha)\frac{z}{1-z} + \alpha\frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty}[1 + (n-1)\alpha z^n].$$

ϕ_α is convex for $|z| < r_\alpha$ and this is sharp and so, for $|z| < r_\alpha$, $\mathrm{Re}\frac{\phi_\alpha(z)}{z} > \frac{1}{2}$. Thus

$$\left(H * \frac{\phi_\alpha}{z}\right) = H + \alpha\frac{zH'}{H} = \left(\frac{k}{4}+\frac{1}{2}\right)\left(h_1 * \frac{\phi_\alpha}{z}\right) - \left(\frac{k}{4}-\frac{1}{2}\right)\left(h_2 * \frac{\phi_\alpha}{z}\right).$$

Using Lemma 2.2, we see that $(h_i * \frac{\phi_\alpha}{z}) \in P$ for $|z| < r_\alpha$, $i = 1, 2$ and hence $(H + \alpha\frac{zH'}{H}) \in P$ for $|z| < r_\alpha$. Consequently, from (3.3), it follows that $f \in M_k(\alpha, \beta)$ for $|z| < r_\alpha$ where r_α is given by (3.2).

Corollary 3.4. *Let $f \in M_k(0, 0)$. Then $f \in M_k(\alpha, 0)$ for $|z| < r_\alpha$, where r_α is given by* (3.2). *That is $f \in R_k$ implies f is of bounded Mocanu variation for $|z| < r_\alpha$.*

Corollary 3.5. *In Corollary* 3.4 *we take $\alpha = 1$. Then it follows that $f \in R_k$ implies $f \in V_k$ for $|z| < \frac{1}{2+\sqrt{3}} = 2 - \sqrt{3}$ and $k = 2$ gives us the radius of convexity for starlike functions.*

References

[1] H. B. Coonce and M. R. Ziegler, "Functions with bounded Mocanu variation," *Rev. Roum. Math. Pures et Appl.*, 19 (1974), 1093-1104.

[2] K. I. Noor, "On a generalization of close-to-convexity," *Int. J. Math. and Math. Sci.*, 6 (1983), 327-334.

[3] B. Pinchuk, "Functions with bounded boundary rotation," *I. J. Math.*, 10 (1971), 7-16.

Mathematics Department, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia.

E-mail: F40M040@KSU.EDU.SA