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WAVE POLYNORMIALS

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Abstract. A generating function for homogeneous polynomial solutions of the wave equation in n-dimensions is obtained. Application is made to developing an integral operator for analytic solutions of the wave equation.

1. Introduction

A homogeneous polynomial of degree k satisfying the wave equation

$$\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\cdots+\frac{\partial^2 u}{\partial x_n^2}-\frac{\partial^2 u}{\partial t^2}=0 \tag{1}$$

is called a homogeneous wave polynomial of degree k in $\mathbb{R}^n$. We present here a generating function for the homogeneous wave polynomials in $\mathbb{R}^n$, $n=2,3,\ldots$. Our result is similar to that given in [2] for the spherical harmonics.

Theorem 1. *The number of linearly independent homogeneous wave polynomials of degree k in $\mathbb{R}^n$ is*

$$d_{n+1}^k=(n+2k-1)\frac{(n+k-2)!}{k!(n-1)!}.$$

Proof. Let P_{n+1}^k denote the vector space of homogeneous polynomials of degree k in $\mathbb{R}^{n+1}$, and note that

$$dim(P_{n+1}^k)=\frac{(n+k)!}{n!k!}.$$

Let W_{n+1}^k denote the subspace of those polynomials which satisfy the wave equation, and let $d_{n+1}^k=dim(W_{n+1}^k)$. Introduce the following inner product on P_{n+1}^k:

$$(f,g)=f\Big(\frac{\partial}{\partial x}\Big)\overline{g},\quad f,g\in P_{n+1}^k.$$

Let $M:P_{n+1}^{k-2}\to P_{n+1}^k$ be given by

$$(Mf)(x_1,\ldots,x_n,t)=(x_1^2+\cdots+x_n^2-t^2)f(x_1,\ldots,x_n,t).$$

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It is easily seen that M is the adjoint of $\Box : P_{n+1}^k \to P_{n+1}^{k-2}$ with respect to the inner product $(\cdot,\cdot)$, where $\Box$ is the wave operator. It follows that

$$P_{n+1}^k = ker(\Box) \oplus Im(M) = W_{n+1}^k \oplus MP_{n+1}^{k-2}.$$

Since M is injective, $dim(MP_{n+1}^{k-2}) = dim(P_{n+1}^{k-2})$. Therefore,

$$\begin{aligned} dim(W_{n+1}^k) &= dim(P_{n+1}^k) - dim(P_{n+1}^{k-2}) \\ &= \frac{(n+k)!}{n!k!} - \frac{(n+k-2)!}{n!(k-2)!} \\ &= (n+2k-1)\frac{(n+k-2)!}{k!(n-1)!}, \end{aligned}$$

which completes the proof.

Note that an elementary argument by induction on k yields

$$d_{n+1}^k = \sum_{j=0}^{k} d_n^j, \quad n = 1,2,3,\ldots.$$

We now show how the homogeneous wave polynomials of degree less than or equal to k in $\mathbb{R}^{n-1}$ can be used to generate the homogeneous wave polynomials of degree k in $\mathbb{R}^n$. This inductive construction is obtained using a simple generating function:

Theorem 2. *If*

$$Q_j(x_1,x_2,\ldots,x_{n-1},t), \quad j = 1,2,\ldots,d_{n+1}^k$$

are linearly independent homogeneous wave polynomials of degree less than or equal to k in $\mathbb{R}^{n-1}$*, and* $s_1^2 + s_2^2 + \cdots + s_n^2 = 1$*, then*

$$(x_1s_1 + x_2s_2 + \cdots + x_ns_n + t)^k = \sum_{j=1}^{d_{n+1}^k} V_k^j(x_1,x_2,\ldots,x_n,t)Q_j(s_1,s_2,\ldots,s_{n-1},is_n)$$

where $V_k^j(x_1,x_2,\ldots,x_n,t)$, $j = 1,2,\ldots,d_{n+1}^k$, *are linearly independent homogeneous wave polynomials of degree k in* $\mathbb{R}^n$.

Proof.

$$(x_1s_1 + x_2s_2 + \cdots + x_ns_n + t)^k = \sum_{v=1}^{l} c_v x^{\alpha_\vee} s^{\alpha_\vee} t^{\beta_\vee}. \tag{2}$$

Here $\alpha_\vee = (\alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_n^\vee)$, and $x^{\alpha_\vee} = x_1^{\alpha_1^\vee} x_2^{\alpha_2^\vee} \cdots x_n^{\alpha_n^\vee}$, and similarly for $t^{\beta_\vee}$ and $s^{\alpha_\vee}$. Since any polynomial of degree j, when restricted to the unit sphere, is a linear combination of spherical harmonics of degree not exceeding j [4], we can express each monomial

$s^{\beta_\vee}$ as a linear combination of the harmonic polynomials $Q_j(s_1, \ldots, s_{n-1}, is_n)$. Thus a rearrangement yields

$$(x_1 s_1 + x_2 s_2 + \cdots + x_n s_n + t)^k = \sum_{j=1}^{d_{n+1}^k} V_k^j(x_1, x_2, \ldots, x_n, t) Q_j(s_1, s_2, \ldots, s_{n-1}, is_n).$$

The V_k^j are clearly homogeneous polynomials of degree k. And since $s_1^2 + s_2^2 + \cdots + s_n^2 = 1$, the linear independence of the $Q_j(s_1, \ldots, s_{n-1}, is_n)$ ensures that each of the polynomials $V_k^j(x_1, x_2, \ldots, x_n, t)$ is a solution of the wave equation (1).

Now we will show that $\{V_k^j\}_{j=1}^{d_{n+1}^k}$ are linearly independent. Since $\{Q_j\}_{j=1}^{d_{n+1}^k}$ are linearly independent, there exist linearly independent homogeneous wave polynomials $\{R_l\}_{l=1}^{d_{n+1}^k}$ such that

$$\int_{\sum_{n-1}} Q_j(s_1, \ldots, s_{n-1}, is_n)\, \overline{R_l(s_1, \ldots, s_{n-1}, is_n)}\, ds = \delta_{jl}.$$

Arguing as before, we have

$$(x_1 u_1 + x_2 u_2 + \cdots + x_n u_n + t)^k = \sum_{l=1}^{d_{n+1}^k} W_k^l(x_1, \ldots, x_n, t) R_l(u_1, \ldots, u_{n-1}, iu_n).$$

Thus

$$\begin{aligned}
&((x_1 s_1 + x_2 s_2 + \cdots + x_n s_n + t)^k, (x_1 u_1 + x_2 u_2 + \cdots + x_n u_n + t)^k) \\
&= \sum_{j=1}^{d_{n+1}^k} \sum_{l=1}^{d_{n+1}^k} (V_k^j W_k^l) Q_j(s_1, \ldots, s_{n-1}, is_n)\, \overline{R_l(u_1, \ldots, u_{n-1}, iu_n)}
\end{aligned}$$

On the other hand, in [5] it was shown that

$$X^k(f^k) = k!(Xf)^k,$$

where X is any first order differential operator for which $X^2 f = 0$. Thus

$$((x_1 s_1 + x_2 s_2 + \cdots + x_n s_n + t)^k, (x_1 u_1 + x_2 u_2 + \cdots + x_n u_n + t)^k) = k!(s_1 u_1 + \cdots + s_n u_n + 1)^k.$$

Further, appealing to the fact that any polynomial of degree j, when restricted to the unit sphere, is a linear combination of spherical harmonics of degree not exceeding j, and using the Funk-Hecke theorem [1, p. 247], it follows that

$$(s_1 u_1 + \cdots + s_n u_n + 1)^k = \sum_{j=1}^{d_{n+1}^k} \lambda_j Q_j(s_1, \ldots, s_{n-1}, is_n) \overline{R_j(u_1, \ldots, u_{n-1}, iu_n)}$$

where the λ_j are non-zero constants.

Thus we have

$$\sum_{j=1}^{d_{n+1}^k}\sum_{l=1}^{d_{n+1}^k}(V_k^j, W_k^l)Q_j(s_1,\ldots,s_{n-1},is_n)\overline{R_l(u_1,\ldots,u_{n-1},iu_n)}$$
$$= k!\sum_{n=1}^{d_{n+1}^k}\lambda_j Q_j(s_1,\ldots,s_{n-1},is_n)\overline{R_l(u_1,\ldots,u_{n-1},iu_n)}$$

from which it follows that $(V_k^j, W_k^l) = \delta_{jl}k!\lambda_j$. Thus, the homogeneous wave polynomials $\{V_k^j\}_{j=1}^{d_{n+1}^k}$ are linearly independent, and this completes the proof.

The generating function given in Theorem 2 can be used to quickly and easily construct all homogeneous wave polynomials up to any given degree in any number of dimensions. The computation is entirely algebraic, and proceeds inductively. In the multinomial expansion (2), one merely expressess the monomials $s^{\beta_\vee}$ as linear combinations of the wave polynomial Q_j. On regrouping, the wave polynomials V_k^j in the next higher dimension appear as coefficients of the wave polynomials Q_j in the lower dimension.

The result of Theorem 2 can also be used to develop an integral operator for analytic solutions of the wave equation (1). First note that the wave polynomials $Q_j(x_1,\ldots,x_{n-1}, t)$ used in the generating function (2) can be chosen so that $Q_j(s_1,\ldots,s_{n-1},is_n)$ are ortho-normal on the unit sphere $\sum_{n-1} : s_1^2 + s_2^2 + \cdots + s_n^2 = 1$. Assume this has been done, and suppose

$$u(x_1,\ldots,x_n,t) = \sum_{h=0}^{\infty}\sum_{j=1}^{d_{n+1}^k} a_{hj}V_h^j(x_1,\ldots,x_n,t)$$

is a solution of the wave equation, where the series of wave polynomials converges uniformly in a neighborhood of the origin. Appealing to the result of Theorem 2, we then have

$$u(x,t) = \sum_{h=0}^{\infty}\sum_{j=1}^{d_{n+1}^k} a_{kj}\int_{\sum_{n-1}}(x\cdot s+t)^k\overline{Q_j(s)}ds$$
$$= \int_{\sum_{n-1}}\sum_{k=0}^{\infty}\sum_{j=1}^{d_{n+1}^k} a_{kj}\overline{Q_j(s)}(x\cdot s+t)^k ds$$
$$= \int_{\sum_{n-1}} f(x\cdot s+t, s)ds.$$

We call f the associate of the wave fucntion u, and write $u = \mathcal{D}_n(f)$. The linear operator $\mathcal{D}_n$ is similar to the integral operator $\mathcal{T}_n$ obtained for harmonic functions in [3], and was first found by Whittaker [6] in the special case $n = 3$.

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