TAMKANG JOURNAL OF MATHEMATICS
Volume 28, Number 3, Autumn 1997

SOME PROPERTIES OF THE FLAG MANIFOLDS

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Abstract. The aim of the present paper is to prove that the read, complex and quaternionic flag manifolds are $\sum$-space.

1. Introduction

Let M be a differentiable manifold and $\sum$ a Lie group which is not required to be connected. If M and $\sum$ have some properties, then M is called $\sum$-space or $\sum$-manifold. One of the problems of $\sum$-manifolds is to determine, if a given manifold can carry $\sum$-structure. ([2]-[7])

The aim of the present paper is to prove that the flag manifolds: $SO(n)/SO(K_1)\times\cdots\times SO(K_v)$, $SU(n)/SU(K_1)\times\cdots\times SU(K_v)$ and $Sp(n)/Sp(K_1)\times\cdots\times Sp(K_v)$ can carry $\sum$-structures.

The whole paper contains five paragraphs.

The second paragraph deals with the general theory of $\sum$-spaces. We also consider special categories of $\sum$-manifolds.

The affine and Riemannian $\sum$-spaces and their different categories are studied in the third paragraph.

In the fourth paragraph we consider the real flag manifold $M = SO(n)/SO(K_1)\times\cdots\times SO(K_v)$, where $K_1+\cdots+K_v = n$, and prove that M is a reduced $\sum$-space. This manifold, with the Riemannian metric, which comes by the restriction on its tangent space m at the origin of the negative of Killing-Cartan form on the Lie algebra $o(n)$, is a reduced Riemannian $\sum$-space.

Finally, in the last paragraph we study the complex and the quaternionic flag manifolds $M_1 = SU(n)/SU(K_1)\times\cdots\times SU(K_v)$ and $M_2 = Sp(n)/Sp(K_1)\times\cdots\times Sp(K_v)$, where $K_1+\cdots+K_v = n$, and prove that they are reduced $\sum$-spaces. The manifolds M_1 and M_2 with the Riemannian metrics, which come by the restriction on m the negative of the Killing-Cartan form on the Lie algebras $u(n)$ and $sp(n)$ respectively become reduced Riemannian $\sum$-spaces.

Received September 17, 1996.
1991 *Mathematics Subject Classification.* 53C30.
Key words and phrases. Lie algebral, Lie group, Killing-cartan form, $\sum$-space and flag monifold.
* This Research is supported by UGC grant No. F:8-13/87(SR-III).

2. Let M be a differentiable manifold, $\sum$ a Lie group and

$$\mu : M \times \sum \times M \to M, \quad \mu : (x, \sigma, y) \to \sigma_x(y)$$

a mooth map. Then the triplet $(M, \sum, \mu)$ is called a $\sum$-space or a $\sum$-manifold, if the following conditions are satisfied

$$\mu(x, \sigma, x) = x \tag{2.1}$$
$$\mu(x, e, y) = y \tag{2.2}$$
$$\mu(x, \sigma, \mu(x, \tau, y)) = \mu(x, \sigma\tau, y) \tag{2.3}$$
$$\mu(x, \sigma, \mu(y, \tau, z)) = \mu(\mu(x, \sigma, y),\ \sigma\tau\sigma^{-1},\ \mu(x, \sigma, z)) \tag{2.4}$$

where $x, y, z \in M$, $\sigma, \tau \in \sum$ and e is the identity element in $\sum$.

From the above we conclude that for each $x \in M$ and $\sigma \in \sum$ a diffeomorphism σ_x on M is defined.

$$\sigma_x : M \to M, \quad \sigma_x : y \to \sigma_x(y) = \mu(x, \sigma, y).$$

and another smooth map σ^x on M is also defined as follow

$$\sigma^x : M \to M, \quad \sigma^x : y \to \sigma^x(y) = \sigma_y(x).$$

The map σ_x satisfies the following conditions

$$\sigma_x(x) = x \tag{2.5}$$

$e_x = id_M$, where e is the identity element of $\sum$ and $\forall \times \in M$ (2.6)

$$\sigma_x \tau_x = (\sigma\tau)_x \tag{2.7}$$

$$\sigma_x \tau_y \sigma_x^{-1} = (\sigma\tau\sigma^{-1})\sigma_x(y) \tag{2.8}$$

For each x we write $\sum_x$ for the image of $\sum$ under the map: $\sum \to \sum_x, : \sigma \to \sigma_x$, then from (2.6) and (2.7) we conclude that $\sum_x$ is a subgroup of Diff(M) and this map is a homomorphism.

For each $\sigma \in \sum$ we define a tensor field S^σ of type (1.1) on the $\sum$-space M as follows

$$S_x^\sigma(X_x) = (\sigma_x) * (X_x) \quad \text{for all } x \in M \text{ and } X_x \in T_x M$$

Then we have the following properties

$$S^\sigma \text{ is smooth} \tag{2.9}$$

$$(\tau_x) * (S^\sigma X) = S^{\tau\sigma\tau^{-1}}((\tau_x) * X),\ \forall \sigma, \tau \in \sum,\ X \in D^1(M),\ x \in M \tag{2.10}$$

$$S^\sigma \text{is Aut(M)-invariant} \tag{2.11}$$

$$(\sigma^x) * X_x = (1 - (\sigma_x)*)X_x = (1 - S^\sigma)X_x \tag{2.12}$$

A $\sum$-space M is called a reduced $\sum$-space, if for each $x \in M$, then T_xM is generated by the set of all $(\sigma^x) * (X_x)$, that is

$$\begin{aligned} T_xM &= gen\{(\sigma^x) * (X_x) : X_x \in T_xM \text{ and } \sigma \in \textstyle\sum\} \\ &= gen\{(\tau - S^\sigma)X_x : X \in T_xM \text{ and } \sigma \in \textstyle\sum\} \end{aligned} \tag{2.13}$$

If $X_x \in T_xM$ and $(\sigma^x) * (X_x) = 0$ for all $\sigma \in \sum$, then $X_x = 0$ and thus no non-zero vector in T_xM is fixed by all $(S^\sigma)_x$ $\forall \sigma \in \sum$ ([3], [4]) (2.14)

3. Now, we consider special structures on $\sum$-spaces.

An affine $\sum$-space is a $\sum$-space M together with an affine connection ∇ with the following property. ([3])

∇ is $\sum_M$-invariant, that means, each σ_x is an affine transformation.

∇ is called canonical, if it also has the property

$$\nabla S^\sigma = 0 \text{ for all } \sigma \in \textstyle\sum$$

A reduced affine $\sum$-space is a reduced $\sum$-space having such connection.

A Riemannian $\sum$-space is a $\sum$-space M together with a $\sum_M$-invariant Riemannian metric g, that means this metric g has the property that each σ_x is an isometry with rescret to the metric g.

A reduced Riemannian $\sum$-space is a reduced $\sum$-space which admits such metric. We shall study only Riemannian reduced $\sum$-spaces. ([3], [4])

4. Now, we consider the real flag manifold ([1])

$$M = SO(n)/SO(K_1) \times \cdots \times SO(K_v), \qquad K_1 + \cdots + K_v = n$$

where $v \geq 3$. If $v = 2$, then the real flag menifold becomes

$$M = SO(n)/SO(K_1) \times SO(K_2), \qquad K_1 + K_2 = n$$

which is called real Grassmann manifold. This is a symmetric space which is a special case of reduced $\sum$-space.

This flag manifold can be written

$$M = G/H$$

where $G = SO(n) = \{A \in GL(n\mathbb{R})/A^tA = I_n\}$ and $H = SO(K_1) \times \cdots \times SO(K_v)$ which consists of matrices of the form

$$H = \left\{ A \in SO(n)/A = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & \ddots \; A_v \end{pmatrix}, \quad A_i \in SO(K_i), \quad i = 1, 2, \ldots, v \right\}.$$

Let g and h be the Lie algebras of G and H respectively. Then we have

$$g = o(n) = \{\alpha \in gl(n, \mathbb{R}) / \alpha +^t \alpha = 0\}$$

and h consists of matrices of the form.

$$h = \left\{ \alpha = \left(\begin{array}{c|c|c|c|c} \alpha_1 & 0 & 0 & \cdots & 0 \\ \hline 0 & \alpha_2 & 0 & \cdots & 0 \\ \hline 0 & 0 & \alpha_3 & \cdots & 0 \\ \hline \cdots & \cdots & \cdots & \cdots & 0 \\ \hline 0 & 0 & 0 & \cdots & \alpha_v \\ \hline & & & & \end{array} \right) / \alpha_i \in o(K_i), \quad i = 1, \ldots, v \right\}.$$

Let m be the tangent space of the flag manifold at its origin 0. Then we have

$$g = h + m.$$

Then m can be represented by matrices as follows

$$m \left\{ \beta = \left(\begin{array}{c|c|c|c|c|c} 0 & -^t X_{12} & -^t X_{13} & \cdots & -^t X_{1v-1} & -^t X_{1v} \\ \hline X_{12} & 0 & -^t X_{23} & \cdots & -^t X_{2v-1} & -^t X_{2v} \\ \hline X_{13} & X_{23} & 0 & \cdots & -^t X_{3v-1} & -^t X_{3v} \\ \hline \cdots & \cdots & \cdots & \cdots & \cdots & \\ \hline X_{1v} & X_{2v} & X_{3v} & \cdots & X_{v-1v} & 0 \end{array} \right) \begin{array}{c} / X_{ij} \in M(K_j \times K_i, \mathbb{R}) \\ \text{the set of matrices} \\ K_j \times K_i \\ 1 \leq i < j \leq v \end{array} \right\}$$

The vector space m can be decomposed as follows

$$m = \oplus m_{ij}$$

$$1 \leq i < j \leq n$$

$$i - \text{column} \quad j - \text{column}$$

$$\begin{array}{c} m_{ij} = \\ i - \text{row} \\ j - \text{row} \end{array} \left(\begin{array}{c|c|c|c|c|c|c} & \cdots & i & \cdots & j & \cdots & 0 \\ \hline & & & & & & \\ \hline & & & & -^t X_{ij} & & \\ \hline & & & & & & \\ \hline & & X_{ij} & & & & \\ \hline & & & & & & \\ \hline & & & & & & 0 \end{array} \right) \tag{4.1}$$

Each vector $\beta \in m$ can be written

$$\beta = \beta_{12} + \cdots + \beta_{1v} + \beta_{23} + \ldots + \beta_{2v} + \ldots + \beta_{v-1v}, \quad \text{where } \beta_{ij} \in m_{ij} \tag{4.2}$$

We consider the matrices

$$\lambda_{\varepsilon_1\varepsilon_2\ldots\varepsilon_v} = \left(\begin{array}{c|c|c|c|c|c} \varepsilon_1 I_{K_1} & 0 & 0 & \cdots & & \\ \hline 0 & \varepsilon_2 I_{K_2} & 0 & \cdots & & \\ \hline 0 & 0 & \varepsilon_3 I_{K_3} & \cdots & & \\ \hline \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ \hline & & & & \varepsilon_{v-1} I_{K_{v-1}} & 0 \\ \hline & & & & 0 & \varepsilon_v I_{K_v} \end{array}\right)$$

where $\varepsilon_i = \pm$, $i = 1, 2, \ldots, v-1, v$. When we have $\varepsilon_i = 1$ we write $\varepsilon_i = i$ and when $\varepsilon_j = -1$, then we write $\varepsilon_j = \bar{j}$ and I_{K_i} $i = 1, 2, \ldots, v$ is the K_i unit matrix. For example:

$$\lambda_{123\ldots v} = \left(\begin{array}{c|c|c|c|c|c} I_{K_1} & & & & & \\ \hline & I_{K_2} & & & & \\ \hline & & I_{K_3} & & & \\ \hline & & & \ddots & & \\ \hline & & & & I_{K_{v-1}} & \\ \hline & & & & & I_{K_v} \end{array}\right),$$

$$\lambda_{\bar{1}\bar{2}\bar{3}4\ldots v} = \left(\begin{array}{c|c|c|c|c|c} -I_{K_1} & & & & & \\ \hline & -I_{K_2} & & & & \\ \hline & & -I_{K_3} & & & \\ \hline & & & \ddots & & \\ \hline & & & & I_{K_{v-1}} & \\ \hline & & & & & I_{K_v} \end{array}\right),$$

$$\lambda_{\bar{1}\bar{2}\bar{3}\ldots\overline{v-1},v} = \left(\begin{array}{c|c|c|c|c|c} -I_{K_1} & & & & & \\ \hline & -I_{K_2} & & & & \\ \hline & & -I_{K_3} & & & \\ \hline & & & \ddots & & \\ \hline & & & & -I_{K_{v-1}} & \\ \hline & & & & & I_{K_v} \end{array}\right),$$

$$\lambda_{\bar{1}\bar{2}\ldots\bar{v}} = \left(\begin{array}{c|c|c|c|c|c} -I_{K_1} & & & & & \\ \hline & -I_{K_2} & & & & \\ \hline & & -I_{K_3} & & & \\ \hline & & & \ddots & & \\ \hline & & & & -I_{K_{v-1}} & \\ \hline & & & & & -I_{K_v} \end{array}\right).$$

The number of such matrices can be found as follows. We consider the following sets

$$J = \{1, 2, 3, \ldots, v\}, \quad \bar{J} = \{\bar{1}, \bar{2}, \bar{3}, \ldots, \bar{v}\}$$

We form the matrices by taking K elements from J and $v - K$ elements from $\bar{J}$, where $K = 0, 1, \ldots, v$. The number of such matrices is

$$\binom{v}{0} + \binom{v}{1} + \binom{v}{2} + \cdots + \binom{v}{v-1} + \binom{v}{v} = 2^v.$$

From each of these matrices we obtain an automorphism on g as follows.

We consider the matrix

$$\lambda_{\bar{1}\bar{2}3\ldots v}$$

and the associated automorphism $\theta_{\bar{1}\bar{2}3\ldots v}$, which is constructed as follows

$$\theta_{\bar{1}\bar{2}3\ldots v} = \lambda_{\bar{1}\bar{2}3\ldots v} \cdot \lambda^{-1}_{\bar{1}\bar{2}3\ldots v} : g \to g$$

$$\theta_{\bar{1}\bar{2}3\ldots v} = \lambda_{\bar{1}\bar{2}3\ldots v} \cdot \lambda^{-1}_{\bar{1}\bar{2}3\ldots v} : \alpha \to \theta_{\bar{1}\bar{2}3\ldots v}(\alpha) = \lambda_{\bar{1}\bar{2}3\ldots v} \cdot \alpha \lambda^{-1}_{\bar{1}\bar{2}3\ldots v}.$$

Proposition 4.1. *For every matrix $\lambda_{\varepsilon_n \ldots \varepsilon_v}$ we can correspond another matrix $\lambda_{\varepsilon'_1 \ldots \varepsilon'_v}$ such that they give the same automorphism on g.*

Proof. Each matrix $\lambda_{\varepsilon_1 \ldots \varepsilon_v}$ has the property

$$\lambda^2_{\varepsilon_1 \ldots \varepsilon_v} = I_n \tag{4.3}$$

which implies

$$\lambda_{\varepsilon_1 \ldots \varepsilon_v} = \lambda^{-1}_{\varepsilon_1 \ldots \varepsilon_v}. \tag{4.4}$$

There exists another matrix $\lambda_{\varepsilon'_1 \varepsilon'_2 \ldots \varepsilon'_v}$ with the property

$$\varepsilon'_1 = (-1)\varepsilon_1, \quad \varepsilon'_2 = (-1)\varepsilon_2, \cdots, \varepsilon'_v = (-1)\varepsilon_v \tag{4.5}$$

and simultaneously satisfies the relation

$$\lambda_{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_v} = -\lambda_{\varepsilon'_1 \varepsilon'_2 \ldots \varepsilon'_n}. \tag{4.6}$$

The associated automorphisms $\theta_{\varepsilon_1 \ldots \varepsilon_v}$ and $\theta_{\varepsilon'_1 \ldots \varepsilon'_v}$ to $\lambda_{\varepsilon_1 \ldots \varepsilon_v}$ and $\lambda_{\varepsilon'_1 \ldots \varepsilon'_v}$ respectively have the form

$$\theta_{\varepsilon_1 \ldots \varepsilon_v} = \lambda_{\varepsilon_1 \ldots \varepsilon_v} \cdot \lambda^{-1}_{\varepsilon_1 \ldots \varepsilon_v} \tag{4.7}$$

$$\theta_{\varepsilon'_1 \ldots \varepsilon'_v} = \lambda_{\varepsilon'_1 \ldots \varepsilon'_v} \cdot \lambda^{-1}_{\varepsilon'_1 \ldots \varepsilon'_v} \tag{4.8}$$

From (4.7) and (4.8) we obtain

$$\theta_{\varepsilon_1 \ldots \varepsilon_v} = \theta_{\varepsilon'_1 \ldots \varepsilon'_v} \tag{4.9}$$

For example

$$\theta_{\bar{1}\bar{2}3\ldots v} = \theta_{12\bar{3}\ldots\bar{v}}, \quad \theta_{\bar{1}\bar{2}\bar{3}\ldots\bar{v}} = \theta_{123\ldots v} \tag{4.10}$$

We denote by T the set of all these automorphisms, that is

$$T = \{\theta_{\varepsilon_1\ldots\varepsilon_v}/\varepsilon_1 = \pm 1, \ldots, \varepsilon_v = \pm 1\} \tag{4.11}$$

Proposition 4.2. *There is an operation on T which makes it a group.*

Proof. We denote by $\cdot$ this operation defined by

$$\cdot : T \times T \to T \tag{4.12}$$

$$\cdot : (\theta_{\varepsilon_1\ldots\varepsilon_v}, \theta_{\varepsilon'_1\ldots\varepsilon'_v}) \to \theta_{\varepsilon_1\ldots\varepsilon_v} \cdot \theta_{\varepsilon'_1\ldots\varepsilon'_v} = \theta_{\varepsilon''_1\ldots\varepsilon''_v} \tag{4.13}$$

where $\theta_{\varepsilon''_1\ldots\varepsilon''_v}$ defined by

$$\varepsilon''_1 = \varepsilon_1 \cdot \varepsilon'_1, \ldots, \varepsilon''_v = \varepsilon_v \cdot \varepsilon'_v \tag{4.14}$$

This operation $\cdot$ turns out the set T onto a finite group with 2^{v-1} elements. The identity element of this group is the automorphism $\theta_{12\ldots v}$. Each element $\theta_{\varepsilon_1\varepsilon_2\ldots\varepsilon_v}$ of this group has the property

$$\theta^2_{\varepsilon_1\ldots\varepsilon_v} = \theta_{1\ldots v}$$

which implies

$$\theta_{\varepsilon_1\ldots\varepsilon_v} = \theta^{-1}_{\varepsilon_1\ldots\varepsilon_v}$$

which means each element of the group T has its inverse the same element.

This group has the following generators

$$\theta_{\bar{1}23\ldots v-1,v}, \theta_{1\bar{2}3\ldots v-1}, \ldots, \theta_{123\ldots,\overline{v-1},v}, \theta_{123\ldots v-1,\bar{v}}$$

Proposition 4.3. *Each element of the group T acts on the Lie algebra $g = h + m$ as follows. It leaves h as fixed pointwise and reverses some of the vectors on m and leaves other fixed. This depends on the form of the automorphism.*

Proof. We take for example the automorphism $\theta_{\bar{1}23\ldots v}$. Therefore we have

$$\theta_{\bar{1}23\ldots v} = \lambda_{\bar{1}23\ldots v} \cdot \lambda^{-1}_{\bar{1}23\ldots v} : h \to h$$

$$\theta_{\bar{1}23\ldots v} = \lambda_{\bar{1}23\ldots v} \cdot \lambda^{-1}_{\bar{1}23\ldots v} : \alpha \to \theta_{\bar{1}23\ldots v}(\alpha) = \lambda_{\bar{1}23\ldots v}\alpha\lambda^{-1}_{\bar{1}23\ldots v}$$

where

$$
\begin{aligned}
&\lambda_{\bar{1}23\ldots v}\alpha\lambda^{-1}_{\bar{1}23\ldots v}\\
&=\left(\begin{array}{c|c|c|c|c}
-I_{K_1} & & & & \\ \hline
 & I_{K_2} & & & \\ \hline
 & & I_{K_3} & & \\ \hline
 & & & \ddots & \\ \hline
 & & & & I_{K_v}
\end{array}\right)
\left(\begin{array}{c|c|c|c|c}
\alpha_1 & & & & \\ \hline
 & \alpha_2 & & & \\ \hline
 & & \alpha_3 & & \\ \hline
 & & & \ddots & \\ \hline
 & & & & \alpha_v
\end{array}\right)
\left(\begin{array}{c|c|c|c|c}
-I_{K_1} & & & & \\ \hline
 & I_{K_2} & & & \\ \hline
 & & I_{K_3} & & \\ \hline
 & & & \ddots & \\ \hline
 & & & & I_{K_v}
\end{array}\right)\\
&=\left(\begin{array}{c|c|c|c|c}
I_{K_1}\alpha_1 I_{K_1} & & & & \\ \hline
 & I_{K_2}\alpha_2 I_{K_2} & & & \\ \hline
 & & I_{K_3}\alpha_3 I_{K_3} & & \\ \hline
 & & & \ddots & \\ \hline
 & & & & I_{K_v}\alpha_v I_{K_v}
\end{array}\right)
=\left(\begin{array}{c|c|c|c|c}
\alpha_1 & & & & \\ \hline
 & \alpha_2 & & & \\ \hline
 & & \alpha_3 & & \\ \hline
 & & & \ddots & \\ \hline
 & & & & \alpha_v
\end{array}\right)=\alpha
\end{aligned}
$$

Hence the automorphism $\theta_{\bar{1}23\ldots v}$ preserves pointwise the subalgebra h of g. The same is true for every other automorphism of T.

We also have

$$
\begin{aligned}
&\theta_{\bar{1}23\ldots v}=\lambda_{\bar{1}23\ldots v}\cdot\lambda^{-1}_{\bar{1}23\ldots v}: m\to m\\
&\theta_{\bar{1}23\ldots v}=\lambda_{\bar{1}23\ldots v}\cdot\lambda^{-1}_{\bar{1}23\ldots v}:\beta\to\theta_{\bar{1}23\ldots v}(\beta)=\lambda_{\bar{1}23\ldots v}\beta\lambda_{\bar{1}23\ldots v}
\end{aligned}
\tag{4.15}
$$

where

$$
\begin{aligned}
&\lambda_{\bar{1}23\ldots v}\cdot\beta\cdot\lambda_{\bar{1}23\ldots v}\\
&=\left(\begin{array}{c|c|c|c|c}
-I_{K_1} & & & & \\ \hline
 & I_{K_2} & & & \\ \hline
 & & I_{K_3} & & \\ \hline
\cdots & \cdots & \cdots & \ddots & \cdots \\ \hline
 & & & & I_{K_v}
\end{array}\right)
\left(\begin{array}{c|c|c|c|c}
0 & -{}^tX_{12} & -{}^tX_{13} & \cdots & -{}^tX_{1v} \\ \hline
X_{12} & 0 & -{}^tX_{23} & \cdots & -{}^tX_{2v} \\ \hline
X_{13} & X_{23} & 0 & \cdots & -{}^tX_{3v} \\ \hline
\cdots & \cdots & \ddots & \cdots & \cdots \\ \hline
X_{1v} & X_{2v} & X_{3v} & \cdots & 0
\end{array}\right)
\left(\begin{array}{c|c|c|c|c}
-I_{K_1} & & & & \\ \hline
 & I_{K_2} & & & \\ \hline
 & & I_{K_3} & & \\ \hline
\cdots & \cdots & \cdots & \ddots & \cdots \\ \hline
 & & & & I_{K_v}
\end{array}\right)\\
&=\left(\begin{array}{c|c|c|c|c}
0 & I_{K_1}{}^tX_{12} & I_{K_1}{}^tX_{13} & \cdots & I_{K_1}{}^tX_{1v} \\ \hline
I_{K_2}X_{12} & 0 & -I_{K_2}{}^tX_{23} & \cdots & -I_{K_2}{}^tX_{2v} \\ \hline
I_{K_3}X_{13} & I_{K_3}X_{23} & 0 & \cdots & -I_{K_3}{}^tX_{3v} \\ \hline
\cdots & \cdots & \ddots & \cdots & \cdots \\ \hline
I_{K_v}X_{1v} & I_{K_v}X_{2v} & I_{K_v}X_{3v} & \cdots & 0
\end{array}\right)\cdot
\left(\begin{array}{c|c|c|c|c}
-I_{K_1} & & & & \\ \hline
 & I_{K_2} & & & \\ \hline
 & & I_{K_3} & & \\ \hline
\cdots & \cdots & \cdots & \ddots & \cdots \\ \hline
 & & & & I_{K_v}
\end{array}\right)\\
&=\left(\begin{array}{c|c|c|c|c}
0 & I_{K_1}{}^tX_{12}I_{K_2} & I_{K_2}{}^tX_{13}I_{K_3} & \cdots & I_{K_1}{}^tX_{1v}I_{K_v} \\ \hline
-I_{K_2}X_{12}I_{K_1} & 0 & -I_{K_2}{}^tX_{23}I_{K_3} & \cdots & -I_{K_2}{}^tX_{2v}I_{K_v} \\ \hline
-I_{K_3}X_{13}I_{K_1} & I_{K_3}X_{23}I_{K_2} & 0 & \cdots & -I_{K_3}{}^tX_{3v}I_{K_v} \\ \hline
\cdots & \cdots & \ddots & \cdots & \cdots \\ \hline
-I_{K_v}X_{1v}I_{K_{1v}} & I_{K_v}X_{2v}I_{K_2} & I_{K_v}X_{3v}I_{K_3} & \cdots & 0
\end{array}\right)
\end{aligned}
$$

$$= \left(\begin{array}{c|c|c|c|c} 0 & {}^tX_{12} & {}^tX_{13} & \cdots & {}^tX_{1v} \\ \hline -X_{12} & & -{}^tX_{23} & \cdots & -{}^tX_{2v} \\ \hline -X_{13} & X_{23} & 0 & & -{}^tX_{2v} \\ \hline \cdots & \cdots & \ddots & \cdots & \cdots \\ \hline -X_{1v} & X_{2v} & X_{3v} & \cdots & \end{array}\right) = \theta_{\bar{1}23\ldots v}(\beta) \tag{4.17}$$

From (4.1), (4.2) and (4.17) we obtain

$$\begin{aligned} &\theta_{\bar{1}23\ldots v}(\beta_{12}) = -\beta_{12}, \theta_{\bar{1}23\ldots v}(\beta_{13}) = -\beta_{13}, \cdots, \theta_{\bar{1}23\ldots v}(\beta_{1v}) = -\beta_{1v}, \\ &\theta_{\bar{1}23\ldots v}(\beta_{23}) = \beta_{23}, \theta_{\bar{1}23\ldots v}(\beta_{24}) = \beta_{24}, \cdots, \theta_{\bar{1}23\ldots v}(\beta_{2v}) = \beta_{2v} \cdots \\ &\theta_{\bar{1}23\ldots v}(\beta_{v-1,v}) = \beta_{v-1,v} \end{aligned} \tag{4.18}$$

Each automorphism $\theta_{\varepsilon_1\varepsilon_2\ldots\varepsilon_v}$ maps the vector β_{ij}, $1 \leq i < j \leq v$ as follows. The automorphism $\theta_{\varepsilon_1\varepsilon_2\ldots\varepsilon_v}$ can be written

$$\theta_{123\ldots,\mu-1,\bar{\mu},\overline{\mu+1},\ldots,\overline{v-1},\bar{v}}$$

where $1 \leq \mu \leq v$. If $i = 1, 2, \ldots, \mu - 1$ and $j = \mu, \mu+1, \ldots, v-1, v$ or the other way, then we have

$$\theta_{\varepsilon_1\varepsilon_2\ldots\varepsilon_v}(\beta_{ij}) = -\beta_{ij} \qquad i < j$$

If $i = 1, 2, \ldots, \mu - 1$ and $j = 2, 3, \ldots, \mu - 1$ $i < j$ or $i = \mu, \mu + 1 \ldots v - 1, v$ and $j = \mu + 1, \ldots, v - 1, v$, then we obtain

$$\theta_{\varepsilon_1\varepsilon_2\ldots\varepsilon_v}(\beta_{ij}) = \beta_{ij} \qquad i < j$$

Theorem 4.4. *The action of the group T on the tangent space m of the flag manifold $M = SO(n)/SO(K_1) \times \cdots \times SO(K_v)$ leaves no vector fixed.*

Proof. Each vector $\beta \in m$ by virtue of (4.2) can be written

$$\beta = \beta_{12} + \beta_{13} + \cdots + \beta_{1v} + \beta_{23} + \beta_{24} + \cdots + \beta_{2v} + \beta_{34} + \cdots + \beta_{v-1v} \tag{4.19}$$

The automorphism $\theta_{\bar{1}2\ldots v}/m$ on m reverses all the vectors

$$\beta_{12}, \beta_{23}, \ldots, \beta_{2v}$$

The automorphism $\beta_{12,\ldots,v-1,\bar{v}}/m$ on m on reverses the vectors

$$\beta_{1v}, \beta_{2v}, \ldots, \beta_{v-1v}$$

From the above we conclude that the automorphisms

$$T_1' = \{\theta_{1\bar{2}3\ldots,v}/m, \theta_{1\bar{2}3\ldots,v}/m, \theta_{123\ldots,v-1,\bar{v}}/m\}$$

on m do not leave vector fixed.

Let T_1 be the set which is defined by $T_1 = T/m \cdot T_1$ is a group with 2^{v-1} elements which have the same properties as the elements of T. T_1 is obtained by the restriction of T on m. Obviously $T_1' C T_1$.

Theorem 4.5. *Let $M = SO(n)/SO(K_1) \times \cdots \times SO(K_v)$ be the real flag manifold. Then M admits a reduced $\sum$-structure, that is, it is a reduced $\sum$-space.*

Proof. We have proved that there is the group T_1 whose each element is a linear transformation on the tangent space m of the homogeneous space M at its origin 0. These linear transformations leave no vector fixed on m.

It is known that every linear transformation

$$\theta_{\varepsilon_1 \ldots \varepsilon_v} : g \to g$$

gives another automorphism on the Lie group G, that is

$$\theta_{\varepsilon_1 \ldots \varepsilon_v} : G \to G$$

which leaves the subgroup H fixed pointwise. This automorphism $\theta_{\varepsilon_1 \ldots \varepsilon_v}$ gives a diffeomorphism $f_{\varepsilon_1 \ldots \varepsilon_v}$ on the manifold $M = G/H$, which is defined by

$$f_{\varepsilon_1 \ldots \varepsilon_v} : M = G/H \to M = G/H$$
$$f_{\varepsilon_1 \ldots \varepsilon_v} : cH \to f_{\varepsilon_1 \ldots \varepsilon_v}(cH) = \theta_{\varepsilon_1 \ldots \varepsilon_v}(c) \cdot H, \quad c \in G$$

From the contruction of $f_{\varepsilon_1 \ldots \varepsilon_v}$ we obtain that

$$f_{\varepsilon_1 \ldots \varepsilon_v} : 0 = H \to f_{\varepsilon_1 \ldots \varepsilon_v}(H) = H = 0$$

that means it fixes 0 and has the property

$$((f_{\varepsilon_1 \ldots \varepsilon_v})*)_0 = \theta_{\varepsilon_1 \ldots \varepsilon_v}$$

The set of automorphisms $\sum_0 = \{f^0_{\bar{1}2,\ldots v}, f^0_{\bar{1}2\ldots v}, \cdots\}$, which is obtained by the group T_1, forms a finite group of 2^{v-1} elements which have the same properties as the elements of T_1.

It is known that the Lie group G acts transitively on the manifold M. This action determines, to every point $x \in M$, a finite group of diffeomorphism on M, $\sum_x = \{f^x_{\bar{1}2\ldots v}, f^x_{\bar{1}2\ldots v} \cdots\}$ with 2^{v-1} elements which have the same properties as the elements of T. Each element of $\sum_x$ is determined as follows. Firstly, we consider the follows mapping:

$$f^0_{\bar{1}2\ldots v} : M \to M, \quad f^0_{\bar{1}2\ldots v} : 0 \to 0$$

It is known that each element of G can be considered as a diffeomorphism on the manifold M. There is one element $\lambda \in G$ with the property

$$\lambda : M \to M, \quad \lambda : 0 \to x \tag{4.20}$$

The element $f^x_{\bar{1}2\dots v}$ of the group S_x which is a diffeormorphism on M, is defined by

$$f^x_{\bar{1}2\dots v} = \lambda o f^o_{\bar{1}2\dots v} o \lambda_{-1} : M \to M$$

This diffeomorphism $f^x_{\bar{1}2\dots v}$ has the property

$$f^x_{\bar{1}2\dots v} : x \to x$$

that means $f^x_{\bar{1}2\dots v}$ leaves x fixed. Therefore the group S_x has the form

$$\sum_x = \{f^x_{12\dots v} = \lambda o f^o_{12\dots v} o \lambda^{-1}, f^x_{\bar{1}2\dots v} = \lambda o f^o_{\bar{1}2\dots v} o \lambda^{-1}, \dots\}$$

Hence to each point $x \in M$ we can associate a finite group of diffeomorphisms $\sum_x$. Each diffeomorphism $f \in \sum_x$ leaves x fixed and $(f_x)*$ for all $f \in \sum_x$ do not leave vector fixed on $T*(M)$. q.e.d.

On the tangent space m of $M = G/H$ at its origin 0 we consider an inner product $<>$ defined by

$$<>: m \times m \to \mathbb{R}, <>: (\beta, \beta') \to <\beta, \beta'> = -\frac{1}{2} Tr(\beta' \beta') \tag{4.21}$$

If K is the Killing-Cartan form on $o(n)$, then we have

$$K : g \times g \to \mathbb{R}, K : (\alpha, \alpha') \to K(\alpha, \alpha') = -(n-1) Tr(\alpha\alpha') \tag{4.22}$$

Since $o(n)$ is a simple Lie algebra, it is known that $K(\alpha, \alpha')$ is negative definite. From (4.21) and (4.22) we concluded that

$$<\beta, \beta> = -\frac{1}{2(n-1)} K(\beta, \beta) \tag{4.23}$$

From (4.23) and the above we conclude that the inner product $<>$ on m is positive definite which gives a Riemannian metric d on M.

Theorem 4.6. *Let $M = SO(n)/SO(K_1) \times \cdots \times SO(K_v)$ be the real flag manifold, where $K_1 + \cdots + K_v = n$. Then the inner product (4.23) on the tangent space m of M at its origin, which comes by the restriction on m of the negative of the Killing-Cartan form on $o(n)$ induces a Riemannian metric d on M. Then (M, d) is a reduced Riemannian $\sum$-space.*

Proof. We assume that the vectors β and β' have the form

$$\beta = \begin{pmatrix} 0 & -{}^tX_{12} & -{}^tX_{13} & \cdots & -{}^tX_{1,v-1} & -{}^tX_{1v} \\ X_{12} & 0 & -{}^tX_{23} & \cdots & -{}^tX_{2,v-1} & -{}^tX_{2v} \\ X_{13} & X_{23} & 0 & \cdots & -{}^tX_{3,v-1} & -{}^tX_{3v} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ X_{1,v-1} & X_{2,v-1} & X_{3,v-1} & \cdots & 0 & -{}^tX_{v-1v} \\ X_{1v} & X_{2v} & X_{3v} & \cdots & X_{v-1v} & 0 \end{pmatrix},$$

$$\beta' = \left(\begin{array}{c|c|c|c|c|c} 0 & -{}^tX'_{12} & -{}^tX'_{13} & \cdots & -{}^tX'_{1v-1} & -{}^tX'_{1v} \\ \hline X'_{12} & 0 & -{}^tX'_{23} & \cdots & -{}^tX'_{2v-1} & -{}^tX'_{2v} \\ \hline X'_{13} & X'_{23} & 0 & \cdots & -{}^tX'_{3v-1} & -{}^tX'_{3v} \\ \hline \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline X'_{1v-1} & X'_{2v-1} & X'_{3v-1} & \cdots & 0 & -{}^tX'_{v-1v} \\ \hline X'_{1v} & X'_{2v} & X'_{3v} & \cdots & X'_{v-1v} & 0 \end{array}\right) \tag{4.24}$$

then we obtain

$\beta\beta' =$

$$\left(\begin{array}{c|c|c} -{}^tX_{12}X'_{12}-\cdots-{}^tX_{1v}X'_{1v} & & \\ \hline & -X_{12}-{}^tX'_{12}-{}^tX_{23}X'_{23}-\cdots-{}^tX_{2v}X'_{2v} & \\ \hline & & -X^t_{13}X'_{13}-X^t_{23}X'_{23}-{}^tX_{34}X'_{34}-\cdots-{}^tX_3X'_3 \\ \hline & & \cdots\cdots\cdots\cdots \\ \hline & & \\ \hline & & \end{array}\right.$$

$$\left.\begin{array}{c|c|c} & & \\ \hline & & \\ \hline & & \\ \hline \ddots & \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots \\ \hline \cdots\cdots\cdots\cdots & -X^t_{1v-1}X'_{1v-1}-\cdots X_{v-2v-1} & -{}^tX_{v-1v}X'_{v-1v} \\ \hline & & -X_{1v}-{}^tX'_{1v}-\cdots-X^t_{v-1v}X_{v-1v} \end{array}\right). \tag{4.25}$$

The relation (4.21) by means of (4.25) implies

$$\begin{aligned}<\beta,\beta'>=&\frac{1}{2}Tr[({}^tX_{12}X'_{12}+\cdots+{}^tX_{1v}X'_{1v})+(X_{12}\,{}^tX'_{12}+{}^tX_{23}X'_{23}+\cdots+{}^tX_{2v}X'_{2v})\\ &+(X_{13}\,{}^tX'_{13}+X_{23}\,{}^tX'_{23}+{}^tX_{34}X'_{34}+\cdots+{}^tX_{3v}X'_{3v})+\cdots+(X_{1v-1}\,{}^tX'_{1v-1}+\cdots\\ &+X_{v-2,v-1}\,{}^tX'_{v-2,v-1}+{}^tX_{v-1v}\,{}^tX'_{v-1v})+(X_{1v}\,{}^tX'_{1v}+\cdots+X_{v-1v}\,{}^tX'_{v-1v})]\end{aligned} \tag{4.26}$$

If $\theta_{\varepsilon_1\ldots\varepsilon_v}$ is one automorphism of the group T_1, then by the properties of $\theta_{\varepsilon_1\ldots\varepsilon_v}$ we have

$$\begin{aligned}&<\theta_{\varepsilon_1\ldots\varepsilon_v}(\beta),\theta_{\varepsilon_1\ldots\varepsilon_v}(\beta')>\\ &=\frac{1}{2}Tr[\theta_{\varepsilon_1\ldots\varepsilon_v}({}^tX_{12})\theta_{\varepsilon_1\ldots\varepsilon_v}(X'_{12})+\cdots+\theta_{\varepsilon_1\ldots\varepsilon_v}({}^tX_{1v})\theta_{\varepsilon_1\ldots\varepsilon_v}(X'_{1v})]\\ &+[\theta_{\varepsilon_1\ldots\varepsilon_v}(X_{12})\theta_{\varepsilon_1\ldots\varepsilon_v}({}^tX'_{12})+\theta_{\varepsilon_1\ldots\varepsilon_v}({}^tX_{23})\theta_{\varepsilon_1\ldots\varepsilon_v}(X'_{23})+\cdots\\ &+\theta_{\varepsilon_1\ldots\varepsilon_v}({}^tX_{2v})\theta_{\varepsilon_1\ldots\varepsilon_v}(X'_{2v})]+[\theta_{\varepsilon_1\ldots\varepsilon_v}(X_{12})\theta_{\varepsilon_1\ldots\varepsilon_v}(X'_{13})\end{aligned}$$

$$+\theta_{\varepsilon_1\ldots\varepsilon_v}({}^tX_{2v})\theta_{\varepsilon_1\ldots\varepsilon_v}(X'_{2v})] + [\theta_{\varepsilon_1\ldots\varepsilon_v}(X_{12})\theta_{\varepsilon_1\ldots\varepsilon_v}(X'_{13})$$
$$+\theta_{\varepsilon_1\ldots\varepsilon_v}(X_{23})\theta_{\varepsilon_1\ldots\varepsilon_v}({}^tX'_{23}) + \theta_{\varepsilon_1\ldots\varepsilon_v}({}^tX_{34})\theta_{\varepsilon_1\ldots\varepsilon_v}(X'_{34}) + \cdots$$
$$+\theta_{\varepsilon_1\ldots\varepsilon_v}({}^tX_{3v})\theta_{\varepsilon_1\ldots\varepsilon_v}(X'_{3v})] + \cdots + [\theta_{\varepsilon_1\ldots\varepsilon_v}(X_{1,v-1})\theta_{\varepsilon_1\ldots\varepsilon_v}({}^tX'_{1,v-1}) + \cdots$$
$$+\theta_{\varepsilon_1\ldots\varepsilon_v}(X_{v-2,v-1})\theta_{\varepsilon_1\ldots\varepsilon_v}({}^tX'_{v-2,v-1}) + \theta_{\varepsilon_1\ldots\varepsilon_v}({}^tX_{v-1v})\theta_{\varepsilon_1\ldots\varepsilon_v}(X'_{v-1v})]$$
$$+[\theta_{\varepsilon_1\ldots\varepsilon_v}(X_{1v})\theta_{\varepsilon_1\ldots\varepsilon_v}({}^tX'_{1v}) + \cdots + \theta_{\varepsilon_1\ldots\varepsilon_v}({}^tX_{v-1v})\theta_{\varepsilon_1\ldots\varepsilon_v}(X'_{v-1v})] \qquad (4.27)$$

We also have

$$\theta_{\varepsilon_1\ldots\varepsilon_v}({}^tX_{ij}) ={}^t X_{ij} \text{ and } \theta_{\varepsilon_1\ldots\varepsilon_v}(X'_{ij}) = X'_{ij}, \quad 1 \le i < j \le v \qquad (4.28)$$

or

$$\theta_{\varepsilon_1\ldots\varepsilon_v}({}^tX_{ij}) = -{}^tX_{ij} \text{ and } \theta_{\varepsilon_1\ldots\varepsilon_v}(X'_{ij}) = -X'_{ij}, \quad 1 \le i < j \le v \qquad (4.29)$$

and

$$\theta_{\varepsilon_1\ldots\varepsilon_v}({}^tX'_{ij}) ={}^t X'_{ij}, \theta_{\varepsilon_1\ldots\varepsilon_v}(X_{ij}) = X_{ij}, \quad 1 \le i < j \le v \qquad (4.30)$$

or

$$\theta_{\varepsilon_1\ldots\varepsilon_v}({}^tX'_{ij}) = -{}^tX'_{ij}, \theta_{\varepsilon_1\ldots\varepsilon_v}(X_{ij}) = -X_{ij}, \quad 1 \le i < j \le v \qquad (4.31)$$

From (4.26), (4.27), (4.28), (4.29), (4.30), and (4.31) we obtain

$$< \beta, \beta' > = < \theta_{\varepsilon_1\ldots\varepsilon_v}(\beta), \theta_{\varepsilon_1\ldots\varepsilon_v}(\beta') >, \quad \forall \theta_{\varepsilon_1\ldots\varepsilon_v} \in T_1 \qquad (4.32)$$

This relation (4.32) completes the proof.

5. We consider the complex flag manifold

$$M_1 = SU(n)/SU(K_1) \times \cdots \times SU(K_v),$$

where $K_1 + \cdots + K_v = n$, $v \ge 3$. If $v = 2$, then the complex flag manifold becomes

$$M_1 = SU(n)/SU(K_1) \times SU(K_2), \quad K_1 + K_2 = n$$

which is called complex Grassmann manifold. This is a symmetric space, which is a special case of a reduced $\sum$-space.

This flag manifold can be written

$$M_1 = G/H$$

where $G = SU(n) = \{A \in GL(n, C)/A * A = I_n\}$ and $H = SU(K_1) \times \cdots \times SU(K_v)$, which consists of matrices of the form

$$H = \left\{ A \in SU(n)/A = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_v \end{pmatrix}, A_i \in SU(K_i), \quad i = 1, \ldots, v \right\}.$$

We consider the Lie algebras g and h of G and H respectively, which are defined by

$$g = u(n) = \{\alpha \in gl(n, C)/\alpha + {}^*\alpha = 0\}$$

$$h = \left\{ \alpha = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_v \end{pmatrix} /\alpha_i \in u(K_i), \quad i = 1, \dots, v \right\}$$

Let m be the tangent space of the complex flag manifold $M_1 = SU(n)/SU(K_1) \times \cdots \times SU(K_v), K_1 + \cdots + K_v = n$, at its origin 0. Then we have

$$g = h + m.$$

Then m can be represented by the matrices as follows

$$m = \left\{ \beta = \begin{pmatrix} 0 & -{}^t\bar{X}_{12} & -{}^t\bar{X}_{13} & & -{}^t\bar{X}_{1,v-1} & -{}^t\bar{X}_{1v} \\ X_{12} & 0 & -{}^t\bar{X}_{23} & & -{}^t\bar{X}_{2,v-1} & -{}^t\bar{X}_{2v} \\ X_{13} & X_{23} & 0 & \cdots & -{}^t\bar{X}_{3,v-1} & -{}^t\bar{X}_{3v} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ X_{1v} & X_{2v} & X_{3v} & \cdots & X_{v-1v} & 0 \end{pmatrix} \begin{matrix} /X_{ij} \in M(K_j \times K_i, \mathbb{C}) \\ \text{the set of matrices} \\ K_j \times K_i/1 \leq i < j \leq v \end{matrix} \right\}$$

We can construct a group $\sum_1$ which has the same properties as $\sum$, defined in §4. Now, we can state the theorem.

Theorem 5.1. *Let $M_1 = SU(n)/SU(K_1) \times \cdots \times SU(K_v)$, $K_1 + \cdots + K_v = n$ be the complex flag manifold. Then M_1 admits a reduced $\sum$-structure, hence it is a reduced $\sum$-space.*

On the tangent space m of $M_1 = SU(n)/SU(K_1) \times \cdots \times SU(K_v)$ at its origin 0 we define the following inner product

$$<,>: m \times m \to \mathbb{R}, <,>: (\beta, \beta') \to <\beta, \beta'>= \frac{1}{2}Tr(\beta\beta') \tag{5.1}$$

If K is the Killing-Cartan form on $u(n)$, then we obtain

$$<,>: g \times g \to \mathbb{R}, K : (\alpha, \alpha') \to K(\alpha, \alpha') = -2nTr(\beta\beta'), \beta = \alpha/m, \beta' = \alpha'/m \tag{5.2}$$

From (5.1) and (5.2) we conclude that

$$<\beta, \beta'>= -\frac{1}{4n}K(\beta, \beta') \tag{5.3}$$

It is known that $K(\beta, \beta') = K(\alpha/m, \alpha'/m)$. Since $K(\alpha, \alpha')$ is negative definite on the simple Lie algebra $u(n)$, we conclude from (5.3) that the inner product $<,>$, defined by (5.1), is positive definite. This inner product defines a metric d_1 on $M_1 = SU(n)/SU(K_1) \times \cdots \times SU(K_v)$, where $K_1 + \cdots + K_v = n$.

Therefore we can state the theorem.

Theorem 5.2. *Let $M_1 = SU(n)/SU(K_1) \times \cdots \times SU(K_v)$ be the complex flag manifold, where $K_1 + \cdots + K_v = n$. Then the inner product (5.1) on the tangent space m of M_1 at its origin, which comes by the restriction on m of the negative of the Killing-Cartan form on $u(n)$ induces a Riemannian metric d_1 on M_1. Then (M_1, d_1) is a reduced Riemannian $\sum$-space.*

Now, we consider the quaternionic flag manifold

$$M_2 = Sp(n)/Sp(K_1) \times \cdots \times Sp(K_v)$$

where $K_1 + \cdots + K_v = n$ $v \geq 3$. If $v = 2$, then quaternionic flag manifold takes the form

$$M_2 = Sp(n)/Sp(K_1) \times Sp(K_2), \qquad K_1 + K_2 = n$$

which is called a quaternionic Grassmann manifold. This is also a symmetric space, which is a special case of a reduced $\sum$-space.

This flag manifold can be written

$$M_2 = G/H$$

where $G = Sp(n) = \{A \in GL(n, \mathbb{H})/A * A = I_n\}$ and $H = Sp(K_1) \times \cdots \times Sp(K_v)$.

It contains matrices of the following form

$$H = \left\{ A \in Sp(n)/A = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_v \end{pmatrix}, A_i \in Sp(K_i), \quad i = 1, \dots v \right\}.$$

Let g and h be the Lie algebras of G and H respectively, which are defined by

$$g = sp(n) = \{\alpha \in gl(n, H)/\alpha^t J + J\alpha = 0\}, \text{ where}$$

$$J = \begin{pmatrix} J_1 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & J_n \end{pmatrix}, \; J_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad i = 1, \dots n$$

$$h = \left\{ \alpha = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_v \end{pmatrix} / \alpha_i \in sp(K_i), \quad i = 1, \dots, v \right\}$$

We consider the tangent space m of the quaternionic flag manifold $M_2 = Sp(n)/Sp(K_1) \times \cdots \times Sp(K_v)$, $K_1 + \cdots + K_v = n$, at its origin 0. Then we have

$$g = h + m$$

The vector space m can be written by form of matrices as follows

$$m = \left\{ \beta = \begin{pmatrix} 0 & -{}^*\bar{X}_{12} & -{}^*\bar{X}_{13} & & -{}^t\bar{X}_{1,v-1} & -{}^t\bar{X}_{1v} \\ X_{12} & 0 & -{}^t\bar{X}_{23} & & -{}^t\bar{X}_{2,v-1} & -{}^t\bar{X}_{2v} \\ X_{13} & X_{23} & 0 & \cdots & -{}^t\bar{X}_{3,v-1} & -{}^t\bar{X}_{3v} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ X_{1v} & X_{2v} & X_{3v} & \cdots & X_{v-1,v} & 0 \end{pmatrix} \begin{array}{l} /X_{ij} \in M(K_j \times K_i, \mathbb{H}) \\ \text{the set of matrices} \\ K_j \times K_i / 1 \leq i < j \leq v \\ \text{and } H \text{ is the field of} \\ \text{quaternionic numbers.} \end{array} \right\}$$

We also can construct a group $\sum_2$ which has the same properties as $\sum$, defined in §4. Now, we have the theorem.

Theorem 5.3. *Let $M_2 = Sp(n)/Sp(K_1) \times \cdots \times Sp(K_v)$, $K_1 + \cdots + K_v = n$, be the quaternionic flag manifold. Then M_2 admits a reduced $\sum$-structure, hence it is a reduced $\sum$-space.*

On the tangent space m of $M_2 = Sp(n)/Sp(K_1) \times \cdots \times Sp(K_v)$ at its origin 0 we define the following inner product

$$<,>: m \times m \to \mathbb{R},\ <,>: (\beta, \beta') \to <\beta, \beta'> = \frac{1}{2} Tr(\beta\beta') \tag{5.4}$$

Let K be the Killing-Cartan form on $Sp(n)$, then we have

$$K : g \times g \to \mathbb{R},\ K : (\alpha, \alpha') \to K(\alpha, \alpha) = -2(n+1) Tr(\beta\beta'), \beta = \alpha/m,\ \beta' = \alpha'/m \tag{5.5}$$

The relations (5.4) and (5.5) imply

$$<\beta, \beta'> = -\frac{1}{4(n+1)} K(\beta, \beta') \tag{5.6}$$

where $K(\beta, \beta') = K(\alpha/m, \alpha'/m)$. Since $K(\alpha, \alpha')$ is negative definite on the simple Lie algebra $Sp(n)$, we obtain from (5.6) that the inner product $<>$, defined by (5.4), is positive definite. This inner product defines a metric d_2 on $M_2 = Sp(n)/Sp(K_1) \times \cdots \times Sp(K_v)$, where $K_1 + \cdots + K_v = n$.

From the above we have the theorem.

Theorem 5.4. *Let $M_2 = Sp(n)/Sp(K_1) \times \cdots \times Sp(K_v)$ be the quaternionic flag manifold, where $K_1 + \cdots + K_v = n$. Then the inner product (5.4) on the tangent space m of M_2 at its origin, which comes by the restriction on m of the negative of the Killing-Cartan form on $Sp(n)$ induces a Riemannian metric d_2 on M_2. Then (M_2, d_2) is a Riemannian $\sum$-space.*

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