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OSCILLATION THEORY FOR A CLASS OF SECOND ORDER QUASILINEAR DIFFERENCE EQUATIONS

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Abstract. In this paper the authors establish necessary and sufficient conditons for the second order quasilinear difference equation

$$\triangle(a_n(\triangle y_n)^{\alpha *}) + f(n, y_{n+1}) = 0, \quad n \in N(n_0) \tag{E}$$

to have various types of nonoscillatory solutions. In addition, in the case when (E) is either strongly superlinear or strongly sublinear, they establish necessary and sufficient conditions for all solutions to oscillate.

1. Introduction

Determining oscillation criteria for difference equations has attracted a great deal of attention in the last several years see, for example [1, 6, 7, 9-11, 13-16] and the references contained therein. In this paper we study the oscillatory and nonoscillatory behavior of solutions of quasilinear difference equations of the type

$$\triangle(a_n(\triangle y_n)^{\alpha_*} + f(n, y_{n+1}) = 0, \quad n \in N(n_0), \tag{1}$$

where $(\triangle y_n)^{\alpha_*} = |\triangle y_n|^\alpha sgn \triangle y_n, N(n_0) = \{n_0, n_0 + 1, \ldots\}$, $n_0 \in \mathbb{N} = \{0, 1, 2, \ldots\}$, $\triangle$ is the forward difference operator defined by $\triangle y_n = y_{n+1} - y_n$, $\{a_n\}$ is a real sequence with $a_n > 0$ for all $n \in N(n_0)$, $\sum_{n=n_0}^{\infty} a_n^{-1/\alpha} < \infty$, $\alpha > 0$ is a constant and $f : N(n_0) \times R \to R$ is continuous, $uf(n, u) > 0$ for $u \neq 0$ and $f(n, u)$ is nondecreasing in u for each fixed $n \in N(n_0)$.

By a solution of equation (1) we mean a nontrivial sequence $\{y_n\}$ satisfying (1) for all $n \in N(n_0)$. A solution $\{y_n\}$ is said to be oscillatory if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise.

First, a classification into three classes of nonoscillatory solutions of equation (1) is given according to their asymptotic behavior as $n \to \infty$, and conditions are obtained for each of the three classes to be nonempty. Next, Criteria are presented for characterising the oscillation of all solutions of equation (1) with the strongly superlinear or strongly

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sublinear condition. The results, in conjunction with the results obtained in [12] for equation (1) with $\{a_n\}$ satisfying $\sum_{n=n_0}^{\infty} a_n^{-1/\alpha} = \infty$, provide a natural generalization of the oscillation theory so far developed for the equation

$$\triangle(a_n \triangle y_n) + f(n, y_{n+1}) = 0$$

beginning with Hooker and Patula [3], Kulenovic and Budincevic [5], He [4], Szmanda [8] and Zhang [17]. Examples which dwell upon the importance of our results are also included.

2. Nonoscillation Theorems

We begin by classifying all possible nonoscillatory solutions of equation (1) according to their asymptotic behavior at infinity. This classification is based on the following Lemma.

Lemma 1. *If $\{y_n\}$ is a nonoscillatory solution of equation* (1), *then there exists positive constants c_1, c_2 and $n_1 \in N(n_0)$ such that*

$$c_1 \eta_\alpha(n) \le |y_n| \le c_2 \quad for \quad n \in N(n_1) \tag{2}$$

where

$$\eta_\alpha(n) = \sum_{s=n}^{\infty} a_n^{-1/\alpha} \tag{3}$$

Proof. Let $\{y_n\}$ be a nonoscillatory solution of equation (1), and without loss of generality, assume that $\{y_n\}$ is eventually positive. Then $\{a_n(\triangle y_n)^{\alpha *}\}$ is eventually decreasing, so $\{\triangle y_n\}$ is eventually of constant sign. If $\triangle y_n > 0$ for $n \in N(n_1)$ for some $n_1 \ge n_0$, then $a_n(\triangle y_n)^\alpha \le a_{n_1}(\triangle y_{n_1})^\alpha$ for $n \in N(n_1)$, so

$$\triangle y_n \le \frac{(a_{n_1})^{1/\alpha} \triangle y_{n_1}}{(a_n)^{1/\alpha}} \quad \text{for } n \in N(n_1).$$

Summing the last inequality from n_1 to $n-1$, we have

$$\begin{aligned} y_n &\le y_{n_1} + (a_{n_1})^{1/\alpha} \triangle y_{n_1} \sum_{s=n_1}^{n-1} (a_s)^{-1/\alpha} \\ &\le y_{n_1} + (a_{n_1})^{1/\alpha} \triangle y_{n_1} \eta_\alpha(n_1), \quad n \in N(n_1). \end{aligned}$$

This proves the right half of the inequality in (2). If $\triangle y_n < 0$ for $n \in N(n_1)$, then, since $a_n(\triangle y_n)^{\alpha *} = -a_n(-\triangle y_n)^\alpha$, we see that

$$a_s(-\triangle y_s)^\alpha \ge a_n(-\triangle y_n)^\alpha, \quad s \ge n \in N(n_1),$$

or

$$-(a_s)^{1/\alpha}\Delta y_s \geq -(a_n)^{1/\alpha}\Delta y_n, \quad s \geq n \in N(n_1). \tag{4}$$

Dividing (4) by $(a_s)^{1/\alpha}$ and summing from n to $j-1$, we have

$$y_n > y_n - y_j \geq -(a_n)^{1/\alpha}\Delta y_n \sum_{s=n}^{j-1}(a_s)^{-1/\alpha}, \quad j > n,$$

which, in the limit as $j \to \infty$, gives

$$y_n \geq -(a_n)^{1/\alpha}\Delta y_n \eta_\alpha(n), \quad n \in N(n_1). \tag{5}$$

Combining (5) with the inequality

$$-(a_n)^{1/\alpha}\Delta y_n \geq -(a_{n_1})^{1/\alpha}\Delta y_{n_1}, \quad n \in N(n_1)$$

which is implied by (4), we find

$$y_n \geq -(a_{n_1})^{1/\alpha}\Delta y_{n_1}\eta_\alpha(n), \quad n \in N(n_1).$$

This proves the left half of the inequality in (2).
A similar argument holds if $\{y_n\}$ is eventually negative.

Lemma 1 shows that the following three types of asymptotic behavior are possible for nonoscillatory solutions $\{y_n\}$ of equation (1):

(I) $\lim_{n\to\infty} y_n = \text{constant} \neq 0$;

(II) $\lim_{n\to\infty} y_n = 0$ and $\lim_{n\to\infty} \frac{|y_n|}{\eta_\alpha(n)} = \infty$;

(III) $\lim_{n\to\infty} \frac{|y_n|}{\eta_\alpha(n)} = \text{constant} > 0$.

First, we characterize the type (I) solutions of equation (1).

Theorem 2. *A necessary and sufficient condition for equation* (1) *to have type* (I) *nonoscillatory solution* $\{y_n\}$ *is that*

$$\sum_{n=n_0}^{\infty}\left(\frac{1}{a_n}\sum_{s=n_0}^{n-1}|f(s,c)|\right)^{1/\alpha} < \infty \tag{6}$$

for some nonzero constant c.

Proof. (Necessity) Let $\{y_n\}$ be a type (I) solution of equation (1) which is positive for $n \in N(n_1)$. Then, there is a constant $c > 0$ such that $y_n > c$ for $n \in N(n_1)$. If $\Delta y_n > 0$ for $n \in N(n_1)$, a summation of equation (1) shows that

$$\sum_{n=n_1}^{\infty} f(n,c) \leq \sum_{n=n_1}^{\infty} f(n, y_{n+1}) \leq a_{n_1}(\Delta y_{n_1})^\alpha < \infty$$

from which (6) follows easily. If $\Delta y_n < 0$ for $n \in N(n_1)$, then summation of (1) gives

$$\sum_{s=n_1}^{n-1} f(s, y_{s+1}) = a_{n_1}(\Delta y_{n_1})^{\alpha_*} - a_n(\Delta y_n)^{\alpha_*} < a_n(-\Delta y_n)^{\alpha}, \quad n \in N(n_1) \tag{7}$$

or

$$\Big(\frac{1}{a_n}\sum_{s=n_1}^{n-1} f(s, y_{s+1})\Big)^{1/\alpha} \le -\Delta y_n, \quad n \in N(n_1).$$

Summing the last inequality from n_1 to j and using the fact that $c_1 \le y_n \le c_2$, $n \in N(n_1)$, for some positive constants c_1 and c_2, we see that

$$\sum_{n=n_1}^{j}\Big(\frac{1}{a_n}\sum_{s=n_1}^{n-1} f(s, c_1)\Big)^{1/\alpha} \le y_{n_1} - y_{j+1} < c_2$$

for any $j > n_1$, which clearly implies (6).

Sufficiency. Suppose that the constant in (6) is positive and $n_2 \ge n_1$ be large enough so that

$$\sum_{n=n_2}^{\infty}\Big(\frac{1}{a_n}\sum_{s=n_2}^{n-1} f(s, c)\Big)^{1/\alpha} \le c/2.$$

Consider the Banach space B_{n_2} of all real sequences $y = \{y_n\}$, $n \in N(n_2)$ with the supremum norm $||y|| = \sup_{n\in N(n_2)} |y_n|$. We define a subset of B_{n_2} as

$$S = \{y \in B_{n_2} : \frac{c}{2} \le y_n \le, \ n \in N(n_2)\}.$$

Clearly, S is a bounded, closed and convex subset of B_{n_2}. Now we define an operator $T : S \to B_{n_2}$ as follows

$$Ty_n = c - \sum_{s=n_2}^{n-1}\Big(\frac{1}{a_s}\sum_{t=n_2}^{s-1} f(t, y_{t+1})\Big)^{1/\alpha}, \quad n \in N(n_2).$$

From the hypothesis this operator T is continuous, and for $y \in S$, we have

$$\frac{c}{2} \le Ty_n \le c \quad \text{for} \quad n \in N(n_2).$$

Thus, $TS \subseteq S$. Therefore, by the Schauder fixed point theorem T has a fixed point $y \in S$. It is clear that $\delta\{y_n\}$ is a positive solution of equation (1) and $\lim_{n\to\infty} y_n$ =constant belongs to $[c/2, c]$. This implies that $\{y_n\}$ is a type (I) solution of equation (1). This completes the proof.

Theorem 3. *A necessary and sufficient condition for equation* (1) *to have a type* (III) *nonoscillatory solution* $\{y_n\}$ *is that*

$$\sum_{n=n_0}^{\infty} |f(n, c\eta_\alpha(n+1)| < \infty \tag{8}$$

for some nonzero constant c.

Proof. (Necessity). Let $\{y_n\}$ be a type (III) solution of equation (1) which is eventually positive. There are constants c_1, $c_2 > 0$ and $n_1 \in N(n_0)$ such that

$$c_1\eta_\alpha(n) \leq y_n \leq c_2\eta_\alpha(n) \quad \text{for} \quad n \in N(n_1).$$

We may suppose that $\Delta y_n < 0$ for $n \in N(n_1)$. From equation (1) we have

$$\sum_{s=n_1}^{n-1} f(s, y_{s+1}) \leq a_n(-\Delta y_n)^\alpha, \quad n \in N(n_1),$$

which is obtained from (7). Combining the last inequality with the inequality

$$a_n(-\Delta y_n)^\alpha \leq \Big(\frac{y_n}{\eta_\alpha(n)}\Big)^\alpha, \quad n \in N(n_1)$$

which is equivalant to (5), we obtain

$$\sum_{s=n_1}^{n-1} f(s, c_1\eta_\alpha(s+1)) \leq \sum_{s=n_1}^{n-1} f(s, y_{s+1}) \leq \Big(\frac{y_n}{\eta_\alpha(n)}\Big)^\alpha \leq c_2^\alpha, \quad n \in N(n_1)$$

which gives (8) in the limit as $n \to \infty$. A similar argument holds if $\{y_n\}$ is an eventually negative solution of type (III) of equation (1).

Sufficiency. We may assume that constant c in (8) is positive. Let $k > 0$ be a constant such that $2k \leq c^\alpha$ and take $n_1 \in N(n_0)$ so that

$$\sum_{n=n_1}^{\infty} f(n, (2k)^{1/\alpha}\eta_\alpha(n+1)) \leq k.$$

Let B_{n_1} be the same Banach space as in the proof of Theorem 2. Define a subset S and an operator $T : S \to B_{n_1}$, by

$$S = \{y \in B_{n_1} : k^{1/\alpha}\eta_\alpha(n) \leq y_n \leq (2k)^{1/\alpha}\eta_\alpha(n), \quad n \in N(n_1)\}$$

$$Ty_n = \sum_{s=n}^{\infty} \Big(\frac{1}{a_s}(k + \sum_{t=n_1}^{s-1} f(t, y_{t+1}))\Big)^{1/\alpha}, \quad n \in N(n_1).$$

As in the proof of Theorem 2, we can show that T satisfies the assumptions of the Schauder fixed point theorem. Hence, there exists $y \in S$ such that $Ty = y$, that is,

$$y_n = \sum_{s=n}^{\infty} \Big(\frac{1}{a_s}(k + \sum_{t=n_1}^{s-1} f(t, y_{t+1}))\Big)^{1/\alpha}, \quad n \in N(n_1).$$

From this equation, it follows that $\{y_n\}$ is a positive solution of equation (1), and that, by Stolz's theorem [2],

$$\lim_{n\to\infty} \frac{y_n}{\eta_\alpha(n)} = \text{constant} \in [k^{1/\alpha}, (2k)^{1/\alpha}].$$

Thus $\{y_n\}$ is a solution of type (III) of equation (1). This completes the proof.

We next give sufficient conditions for the existence of a type (II) solution of equation (1).

Theorem 4. *The equation* (1) *has a nonoscillatory solution of type* (II) *if*

$$\sum_{n=n_0}^{\infty}\left(\frac{1}{a_n}\sum_{s=n_0}^{n-1}|f(s,c)|\right)^{1/\alpha}<\infty \tag{9}$$

for some nonzero constant c and

$$\sum_{n=n_0}^{\infty}|f(n,d\eta_\alpha(n+1)|=\infty \tag{10}$$

for any constant $d \neq 0$ with $cd > 0$.

Proof. Suppose that the c in (9) is positive. Choose $k > 0$ such that $(2k)^{1/\alpha}(\eta_\alpha(n_0)+1) \leq c$ and let $n_1 \in N(n_0)$ be large enough so that

$$\sum_{n=n_1}^{\infty}\left(\frac{1}{a_n}\sum_{s=n_1}^{n-1}f(s,c)\right)^{1/\alpha}\leq k^{1/\alpha}.$$

Let B_{n_1} be the same as in the proof of Theorem 3, and define a subset S and an operator $T : S \to B_{n_1}$ by

$$S=\{y\in B_{n_1}: k^{1/\alpha}\eta_\alpha(n)\leq y_n\leq c,\quad n\in N(n_1)\}$$
$$Ty_n=\sum_{s=n}^{\infty}\left(\frac{1}{a_s}(k+\sum_{t=n_1}^{s-1}f(t,y_{t+1}))\right)^{1/\alpha},\quad n\in N(n_1).$$

Similar to the proof of Theorem 2, we can show that the mapping satisfies the assumptions of Schauder fixed point theorem. Hence there exists $y \in S$ such that $Ty = y$; that is, $\{y_n\}$ is a nonoscillatory solution of equation (1) satisfying $\lim_{n\to\infty} y_n = 0$. To see that $\frac{y_n}{\eta_\alpha(n)} \to \infty$ as $n \to \infty$, we use Stolz's theorem [2] and (10) to obtain

$$\lim_{n\to\infty}\frac{y_n}{\eta_\alpha(n)}=\lim_{n\to\infty}\frac{\Delta y_n}{\Delta\eta_\alpha(n)}=\lim_{n\to\infty}\left(k+\sum_{s=n_1}^{n-1}f(s,y_{s+1})\right)^{1/\alpha}$$
$$\geq\lim_{n\to\infty}\left(k+\sum_{s=n_1}^{n-1}f(s,k^{1/\alpha}\eta_\alpha(s+1))\right)^{1/\alpha}=\infty.$$

Example 1. Consider the difference eqution

$$\Delta(2^{\lambda_n}(\Delta y_n)^{\alpha_*})+2^{\mu n}y_{n+1}^{\beta_*}=0,\quad n\in\mathbb{N} \tag{11}$$

where $y_{n+1}^{\beta *} = |y_{n+1}|^{\beta} sgn\, y_{n+1}$, and $\alpha > 0$, $\beta > 0$, $\lambda > 0$ and μ are constants. Applying Theorem 2 and 3 to this equation, we see that:

(i) Equation (11) possesses a nonoscillatory solution of type (I) if and only if $\mu < \lambda$;

(ii) Equation (11) possesses a nonoscillatory solution of type III (which behaves like a constant multiple of $2^{(-\lambda/\alpha)n}$ as $n \to \infty$) if and only if $\mu/\lambda < \beta/\alpha$.

Theorem 4 implies that equation (11) possesses a nonoscillatory solution of type (II) if $\beta/\alpha < \mu/\lambda < 1$.

3. Oscillation Theorems

In this section we study the oscillatory behavior of solutions of equation (1). In view of the results of Hooker ane Patula [3] and Zhang [17], it is reasonable to expect that a characterization of oscillation for equation (1) can be obtained under suitable additional conditions on the nonlinear funciton f.

Definition 5. (i) The equation (1) (or the function $f(n,u)$) is said to be strongly superlinear if there is a constant $\gamma > \alpha$ such that $|u|^{-\gamma} f(n,u)$ is nondecreasing in u for each fixed $n \in N(n_0)$.

(ii) The equation (1) (or the function $f(n,u)$) is said to be strongly sublinear if there is a positive constant $\gamma < \alpha$ such that $|u|^{-\gamma} f(n,u)$ is nonincreasing in u for each fixed $n \in N(n_0)$.

Theorem 6. *Let equation* (1) *be strongly superlinear. All solutions of equation* (1) *are oscillatory if and only if*

$$\sum_{n=n_0}^{\infty} |f(n, c\eta_\alpha(n+1)| = \infty \tag{12}$$

for every nonzero constant c.

Proof. The necessary part follows from Theorem 3. To prove sufficiency suppose that equation (1) has a nonoscillatory solution $\{y_n\}$, say $y_n > 0$ for $n \geq n_1 \in N(n_0)$. Then, either $\Delta y_n > 0$ for $n \in N(n_1)$ or $\Delta y_n < 0$ for $n \in N(n_1)$.

In the first case, it is easy to see that $\sum_{n=n_1}^{\infty} f(n, c_1) < \infty$ for some $c_1 > 0$. Since $\eta_\alpha(n) \to 0$ as $n \to \infty$, there exists an integer $n_2 \in N(n_1)$ such that $y_n \geq c_1 \eta_\alpha(n)$ for $n \in N(n_2)$. Hence

$$\sum_{n=n_2}^{\infty} f(n, c_1\eta_\alpha(n+1)) < \infty,$$

which contradicts (12). If $\Delta y_n < 0$ for $n \in N(n_1)$, then by Lemma 1, there is a constant $c_2 > 0$ such that $y_n \geq c_2 \eta_\alpha(n)$ for $n \in N(n_1)$.

By the strong superlinearity of f, we have

$$y_{n+1}^{-\gamma} f(n, y_{n+1}) \geq (c_2\eta_\alpha(n+1))^{-\gamma} f(n, c_2\eta_\alpha(n+1)), \quad n \in N(n_1) \tag{13}$$

for some constant $\gamma > \alpha$. We see the difference

$$\triangle\{-[-a_n(\triangle y_n)^{\alpha_*}]^{-(\gamma-\alpha)/\alpha}\} = \frac{\gamma-\alpha}{\alpha}t^{-\gamma/\alpha}f(n, y_{n+1}) \tag{14}$$

where $-a_{n+1}(\triangle y_{n+1})^{\alpha_*} < t < -a_n(\triangle y_n)^{\alpha_*}$. Using (13) and (5) in (14) we have

$$\begin{aligned}\triangle\{-[-a_n(\triangle y_n)^{\alpha_*}]^{-(\gamma-\alpha)/\alpha}\} &\geq \frac{\gamma-\alpha}{\alpha}[-a_{n+1}(\triangle y_{n+1})^{\alpha_*}]^{-\gamma/\alpha}y_{n+1}^{\gamma}y_{n+1}^{-\gamma}f(n, y_{n+1}) \\ &\geq \frac{\gamma-\alpha}{\alpha}(-a_{n+1}(\triangle y_{n+1})^{\alpha_*})^{-\gamma/\alpha}(-a_{n+1}(\triangle y_n)^{\alpha_*})^{\gamma/\alpha}X \\ &\quad \eta_\alpha^{\gamma}(n+1)(c_2\eta_\alpha(n+1))^{-\gamma}f(n, c_2\eta_\alpha(n+1)) \\ &= \frac{\gamma-\alpha}{\alpha}c_2^{-\gamma}f(n, c_2\eta_\alpha(n+1)), \quad n \in N(n_1).\end{aligned}$$

Summation of the above yields

$$\frac{\gamma-\alpha}{\alpha}c_2^{-\gamma}\sum_{s=n_1}^{n} f(s, c_2\eta(s+1)) \leq [-a_n(\triangle y_{n_1})^{\alpha_*}]^{-(\gamma-\alpha)/\alpha}, \quad n \in N(n_1)$$

which implies that $\sum\limits_{n=n_1}^{\infty} f(n, c_2\eta_\alpha(n+1)) < \infty$, again contradicting (12). This completes the proof of the theorem.

Theorem 7. *Let equation* (1) *be strongly sublinear. All solutions of equation* (1) *are oscillatory if and only if*

$$\sum_{n=n_0}^{\infty}\Big(\frac{1}{a_n}\sum_{s=n_0}^{n-1}|f(s,c)|^{1/\alpha}\Big) = \infty \tag{15}$$

for every nonzero constant c.

Proof. It is enough to prove sufficient part, since the necessary part follows from Theorem 2. Let condition (15) holds and equation (1) has a nonoscillatory solution $\{y_n\}$ which is positive for $n \in N(n_0)$.

If $\triangle y_n > 0$ for $n \geq n_1 \in N(n_0)$, then there is a constant $c_1 > 0$ such that $\sum_{n=n_1}^{\infty} f(s, c_1) < \infty$, and this together with the condition $\sum\limits_{n=n_1}^{\infty} a_n^{-1/\alpha} < \infty$ shows that

$$\sum_{n=n_1}^{\infty}\Big(\frac{1}{a_n}\sum_{s=n_1}^{n-1} f(s, c_1)\Big)^{1/\alpha} \leq \eta_\alpha(n_1)\Big(\sum_{n=n_1}^{\infty} f(s, c_1)\Big) < \infty,$$

which contradicts (15).

If $\triangle y_n < 0$ for $n \in N(n_1)$, then summation of equation (1) gives $\sum\limits_{s=n_1}^{n-1} f(s, y_{s+1}) \leq a_n(-\triangle y_n)^{\alpha}$, $\quad n \in N(n_1)$ (cf. (7)), or equivalents

$$-\triangle y_n \geq \Big(\frac{1}{a_n}\sum_{s=n_1}^{n-1} f(s, y_{s+1})\Big)^{1/\alpha}, \quad n \in N(n_1) \tag{16}$$

Since $y_n \leq c_2$ for $n \in N(n_1)$, for some constants $c_2 > 0$, the strongly sublinearity implies

$$y_{n+1}^{-\partial} f(n, y_{n+1}) \geq c_2^{-\partial} f(n, c_2), \quad n \in N(n_1) \tag{17}$$

for some $\partial < \alpha$. Combining (16) with (17) and using the decreasing property of $\{y_n\}$, we see that

$$\begin{aligned} -\Delta y_n &\geq c_2^{-\partial/\alpha} a_n^{-1/\alpha} \Big(\sum_{s=n_1}^{n-1} y_{s+1}^{\partial} f(s, c_2) \Big)^{1/\alpha} \\ &\geq c_2^{-\partial/\alpha} a_n^{-1/\alpha} y_{n+1}^{\partial/\alpha} \Big(\sum_{s=n_1}^{n-1} f(s, c_2) \Big)^{1/\alpha}, \quad n \in N(n_1). \end{aligned} \tag{18}$$

Now, we consider the difference $\Delta(-(y_n)^{\alpha-\partial/\alpha})$;

$$\Delta(-(y_n)^{\alpha-\partial/\alpha}) = -\frac{(\alpha-\partial)}{\alpha} t^{-\partial/\alpha} \Delta y_n \tag{19}$$

where $y_{n+1} < t < y_n$ and $\{y_n\}$ is decreasing.
From (18) and (19) we have

$$\begin{aligned} \Delta(-(y_n)^{\alpha-\partial/\alpha}) &\geq \frac{\alpha-\partial}{\alpha} c_2^{-\partial/\alpha} a_n^{-1/\alpha} y_{n+1}^{\partial/\alpha} \Big(\sum_{s=n_1}^{n-1} f(s, c_2) \Big)^{1/\alpha} y_{n+1}^{-\partial/\alpha} \\ &= \frac{\alpha-\partial}{\alpha} c_2^{-\partial/\alpha} a_n^{-1/\alpha} \Big(\sum_{s=n_1}^{n-1} f(s, c_2) \Big)^{1/\alpha}, \quad n \in N(n_1). \end{aligned}$$

Summation of last inequality yields

$$\frac{\alpha-\partial}{\alpha} c_2^{-\partial/\alpha} \sum_{s=n_1}^{n-1} \Big(\frac{1}{a_s} \sum_{i=n_1}^{s-1} f(i, c_2) \Big)^{1/\alpha} \leq (y_{n_1})^{(\alpha-\partial)/\alpha}$$

for $n \in N(n_1)$, which again contradicts (15). This complete the proof.

Example 2. Consider the difference equation

$$\Delta(2^{\lambda n}(\Delta y_n)^{\alpha_*}) + 2^{\mu n} y_{n+1}^{\beta_*} = 0, \quad n \in \mathbb{N} \tag{20}$$

which is the same as (11).

Let $\alpha < \beta$. Then, by Theorem 6 all solutions of (20) are oscillatory if and only if $\mu/\lambda \geq \beta/\alpha$. Let $\alpha > \beta$. Then, by Theorem 7 all solutions of (20) are oscillatory if and only if $\mu \geq \lambda$.

Remark. The results of this paper generalize some of the results of He [4] and Zhang [17]. Also the results of this paper generalize some of the results obtained in [10] in the sense that here we do not require the condition $a_n \to \infty$ as $n \to \infty$.

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