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SOME REFINEMENTS OF HADAMARD'S INEQUALITIES

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Abstract. Some new refinement of Hadamard's inequalities are given.

1. Introduction

If $f : [a,b] \to R$ is a convex function, then

$$f\Big(\frac{a+b}{2}\Big) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1}$$

are known in the literature as Hadamard's inequalities.

In [1], S. S. Dragomir generalized (1) into the following:

Theorem A. *Let $f : [a,b] \to R$ be a convex function. Then for all $t \in [0,1]$ we have the following inequalities:*

$$f\Big(\frac{a+b}{2}\Big) \le \frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)dxdy \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}.$$

In [4], S. S. Dragomir, J. E. Pecaric and J. Sandor established a further refinement of the first inequality of (1) as following:

Theorem B. *Let $f : [a,b] \to R$ be a convex function and let n be a natural number with $n \ge 2$. Then*

$$\begin{aligned} f\Big(\frac{a+b}{2}\Big) &\le \frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b f\Big(\frac{x_1+x_2+\cdots+x_n}{n}\Big)dx_1dx_2\cdots dx_n \\ &\le \frac{1}{(b-a)^{n-1}}\int_a^b \cdots \int_a^b f\Big(\frac{x_1+x_2+\cdots+x_{n-1}}{n-1}\Big)dx_1dx_2\cdots dx_{n-1} \\ &\le \frac{1}{b-a}\int_a^b f(x)dx. \end{aligned} \tag{2}$$

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Theorem B has been generalized by S. S. Dragomir and C. Buse (see [3]) as follows.

Theorem C. *Let $f : [a,b] \to R$ be a convex continuous function, and let $q_i \geq 0$ $(i = 1, 2, \ldots, n)$ with $Q_n = \sum_{i=1}^{n} q_1 > 0$. Then*

$$\begin{aligned} f\Big(\frac{a+b}{2}\Big) &\leq \frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b f\Big(\frac{x_1+x_2+\cdots+x_n}{n}\Big)dx_1dx_2\cdots dx_n \\ &\leq \frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b f\Big(\frac{q_1x_1+q_2x_2+\cdots+q_nx_n}{Q_n}\Big)dx_1dx_2\cdots dx_n \\ &\leq \frac{1}{b-a}\int_a^b f(x)dx. \qquad (3) \end{aligned}$$

We note that if $q_i = \frac{1}{n-1}$ for $i = 1, 2, \ldots, n-1$ and $q_n = 0$, then (3) reduce to (2). Also, we note that both

$$\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b f\Big(\frac{q_1x_1+q_2x_2+\cdots+q_nx_n}{Q_n}\Big)dx_1dx_2\cdots dx_n$$

in (3) and

$$\frac{1}{(b-a)^{n-1}}\int_a^b \cdots \int_a^b f\Big(\frac{x_1+x_2+\cdots+x_{n-1}}{n-1}\Big)dx_1dx_2\cdots dx_{n-1}$$

in (2) lie between $\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b f\Big(\frac{x_1+x_2+\cdots+x_n}{n}\Big)dx_1dx_2\cdots dx_n$ and $\frac{1}{b-a}\int_a^b f(x)dx$, but they are not comparable. For instance, let $f(x) = x^2$. Then

$$\begin{aligned} \int_0^1\int_0^1\int_0^1 \Big(\frac{2x_1+x_2+x_3}{4}\Big)^2 dx_1dx_2dx_3 &= \frac{9}{32} \\ &< \frac{7}{24} = \int_0^1\int_0^1 \Big(\frac{x_1+x_2}{2}\Big)^2 dx_1dx_2 \\ &< \frac{115}{384} = \int_0^1\int_0^1\int_0^1 \Big(\frac{6x_1+x_2+x_3}{8}\Big)^2 dx_1dx_2dx_3. \end{aligned}$$

Recently, S. S. Dragomir [2] constructed a convex increasing function which lies between the first inequality of (1) as follows.

Theorem D. *Let $f : [a,b] \to R$ be a convex function, and let $H : [0,1] \to R$ be defined by*

$$H(t) = \frac{1}{b-a}\int_a^b f\Big(tx + (1-t)\frac{a+b}{2}\Big)dx.$$

Then (i) H is convex on [0,1],
(ii) $\inf_{t\in[0,1]} H(t) = H(0) = f\Big(\frac{a+b}{2}\Big)$ and $\sup_{t\in[0,1]} H(t) = H(1) = \frac{1}{b-a}\int_a^b f(x)dx$,
(iii) H increases monotonically on [0,1].
The main purpose of this note is to generalize Theorem A, Theorem B and Theorem D.

2. Main Results

First, we have the following theorem:

Theorem 1. *Let* $f : [a,b] \to R$ *be a convex function, and* $0 < \alpha_i < 1$ $(i = 1, 2, \ldots, n; n \geq 2)$ *with* $\sum_{i=1}^{n} \alpha_i = 1$. *Then*

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b f(\alpha_1 x_1 + \cdots + \alpha_n x_n)dx_1 dx_2 \cdots dx_n$$
$$\leq \sum_{i=1}^{n} \frac{1-\alpha_i}{n-1}\frac{1}{(b-a)^{n-1}}\int_a^b \cdots \int_a^b f\Big(\frac{1}{1-\alpha_i}\sum_{j=1,j\neq i}^{n} \alpha_j x_j\Big)dx_1 \cdots dx_{i-1}dx_{i+1}\cdots dx_n$$
$$\leq \frac{1}{b-a}\int_a^b f(x)dx. \tag{4}$$

Proof. By Jensen's inequality, we have

$$f\Big(\frac{a+b}{2}\Big) = f\Big(\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b (\alpha_1 x_1 + \cdots + \alpha_n x_n)dx_1 dx_2 \cdots dx_n\Big)$$
$$\leq \frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b f(\alpha_1 x_1 + \cdots + \alpha_n x_n)dx_1 dx_2 \cdots dx_n.$$

This proves the first inequality of (4).
Now,

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = \frac{1}{n-1}\sum_{i=1}^{n}\Big(\sum_{j=1,j\neq i}^{n} \alpha_j x_j\Big) = \sum_{i=1}^{n}\frac{1-\alpha_i}{n-1}\Big(\frac{1}{1-\alpha_i}\sum_{j=1,j\neq i}^{n} \alpha_j x_j\Big).$$

Since $\sum_{i=1}^{n} \frac{1-\alpha_i}{n-1} = 1$ and $\frac{1}{1-\alpha_i}\sum_{j=1,j\neq i}^{n} \alpha_j = 1$, by repeated using the convexity of f, we obtain

$$f(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n) \leq \sum_{i=1}^{n}\frac{1-\alpha_i}{n-1} f\Big(\frac{1}{1-\alpha_i}\sum_{j=1,j\neq i}^{n} \alpha_j x_j\Big) \tag{5}$$
$$\leq \sum_{i=1}^{n}\frac{1}{n-1}\Big[\sum_{j=1,j\neq i}^{n} \alpha_j f(x_j)\Big] = \sum_{i=1}^{n} \alpha_i f(x_i). \tag{6}$$

Hence, using (5), we have

$$\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b f(\alpha_1 x_1 + \cdots + \alpha_n x_n)dx_1 dx_2 \cdots dx_n$$
$$\leq \frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \sum_{i=1}^{n}\frac{1-\alpha_i}{n-1} f\Big(\frac{1}{1-\alpha_i}\sum_{j=1,j\neq i}^{n} \alpha_j x_j\Big)dx_1 dx_2 \cdots dx_n$$

$$= \sum_{i=1}^{n} \frac{1-\alpha_i}{n-1} \frac{1}{(b-a)^n} \int_a^b \cdots \int_a^b f\Big(\frac{1}{1-\alpha_i} \sum_{j=1, j\neq i}^{n} \alpha_j x_j\Big) dx_1 dx_2 \cdots dx_n$$

$$= \sum_{i=1}^{n} \frac{1-\alpha_i}{n-1} \frac{1}{(b-a)^{n-1}} \int_a^b \cdots \int_a^b f\Big(\frac{1}{1-\alpha_i} \sum_{j=1, j\neq i}^{n} \alpha_j x_j\Big) dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n.$$

This proves the second inequality of (4).
Finally, using (6), we have

$$\sum_{i=1}^{n} \frac{1-\alpha_i}{n-1} \frac{1}{(b-a)^{n-1}} \int_a^b \cdots \int_a^b f\Big(\frac{1}{1-\alpha_i} \sum_{j=1, j\neq i}^{n} \alpha_j x_j\Big) dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n$$

$$\leq \frac{1}{(b-a)^n} \int_a^b \cdots \int_a^b \sum_{i=1}^{n} \frac{1}{n-1} \Big[\sum_{j=1, j\neq i}^{n} \alpha_j f(x_j) \Big] dx_1 dx_2 \cdots dx_n$$

$$= \frac{1}{(b-a)^n} \int_a^b \cdots \int_a^b \Big[\sum_{i=1}^{n} \alpha_i f(x_i) \Big] dx_1 dx_2 \cdots dx_n$$

$$= \frac{1}{(b-a)} \int_a^b f(x) dx.$$

This completes the proof.

Remark 1. In case $n = 2$ and $\alpha_1 = t, \alpha_2 = 1-t$, we have

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{(b-a)^2} \int_a^b \int_a^b f(tx + (1-t)y) dx dy \leq \frac{1}{b-a} \int_a^b f(x) dx$$

for $t \in [0,1]$, which is Theorem A.

Remarks 2. Theorem B is the special case of (3) when $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \frac{1}{n}$.

Further, we constract a convex, increasing function between the first inequality in (4) as follows.

Theorem 2. *Let $f : [a,b] \to R$ be a convex function, and $0 < \alpha_i < 1$ $(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} \alpha_i = 1$. If $K : [0,1] \to R$ is a function defined by*

$$K(t) = \frac{1}{(b-a)^n} \int_a^b \int_a^b f\Big(t \sum_{i=1}^{n} \alpha_i x_i + (1-t) \frac{a+b}{2}\Big) dx_1 dx_2 \cdots dx_n.$$

then

(i) *K is convex on [0,1],*

(ii) $f\Big(\frac{a+b}{2}\Big) = K(0) = \min_{t\in[0,1]} K(t) \leq K(t) \leq \max_{t\in[0,1]} K(t) = K(1)$

$$= \frac{1}{(b-a)^n} \int_a^b \cdots \int_a^b f\Big(\sum_{i=1}^{n} \alpha_i x_i\Big) dx_1 dx_2 \cdots dx_n \text{ for all } t \in [0,1],$$

(iii) *K is increasing on [0,1].*

Proof. For $t \in [0,1]$, let $h(t) = t\sum_{i=1}^{n} \alpha_i x_i + (1-t)\frac{a+b}{2}$. Then h is linear so that the composition function $f \circ h$ is convex on [0,1], which implies that K is convex on [0,1]. This proves (i).

Next, using the convexity of f and the first inequality of Theorem 1, we have

$$
\begin{aligned}
K(t) &\leq \frac{t}{(b-a)^n}\int_a^b \cdots \int_a^b f\Big(\sum_{i=1}^{n} \alpha_i x_i\Big) dx_1 dx_2 \cdots dx_n \\
&\quad +(1-t)\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b f\Big(\frac{a+b}{2}\Big) dx_1 dx_2 \cdots dx_n \\
&= \frac{t}{(b-a)^n}\int_a^b \cdots \int_a^b f\Big(\sum_{i=1}^{n} \alpha_i x_i\Big) dx_1 dx_2 \cdots dx_n + (1-t) f\Big(\frac{a+b}{2}\Big) \\
&\leq \frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b f\Big(\sum_{i=1}^{n} \alpha_i x_i\Big) dx_1 dx_2 \cdots dx_n \\
&= K(1).
\end{aligned}
$$

Now, by Jensen's inequality, we have

$$
\begin{aligned}
K(t) &\geq f\Big(\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \Big[t\sum_{i=1}^{n} \alpha_i x_i + (1-t)\frac{a+b}{2}\Big] dx_1 dx_2 \cdots dx_n\Big) \\
&= f\Big(t\sum_{i=1}^{n} \frac{\alpha_i}{b-a}\int_a^b x_i dx_i + (1-t)\frac{a+b}{2}\Big) \\
&= f\Big(t\sum_{i=1}^{n} \alpha_i (\frac{a+b}{2}) + (1-t)\frac{a+b}{2}\Big) \\
&= f\Big(\frac{a+b}{2}\Big) = K(0).
\end{aligned}
$$

This completes the proof of (ii).

Finally, using the convexity of K and $K(t) \geq K(0)$, we have

$$
\frac{K(u)-K(t)}{u-t} \geq \frac{K(t)-K(0)}{t} \geq 0, \qquad \text{if} \qquad 0 \leq t < u \leq 1,
$$

so that $K(t) \leq K(u)$.

This completes the proof of (iii).

Remark 4. We note that Theorem D is the special case of Theorem 2 when $n=1$.

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