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EXISTENCE AND UNIQUENESS OF MILD AND STRONG SOLUTIONS OF A VOLTERRA INTERGRODIFFERENTIAL EQUATION WITH NONLOCAL CONDITIONS

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Abstract. We prove the existence and uniqueness of mild and strong solutions of a Volterra integrodifferential equation with nonlocal initial conditions using the method of semigroups and the Banach fixed point theorem.

1. Introduction

Using the method of semigroups and the Banach fixed point theorem, Byszewski [2] studied about the existence and uniqueness of mild, strong and classical solutions of the following nonlocal Cauchy problem:

$$\frac{du(t)}{dt} + Au(t) = f(t, u(t)), \quad t \in (t_0, t_0 + a]$$

$$u(t_0) + g(t_1, \ldots, t_p, u(\cdot)) = u_0,$$

where $-A$ is the infinitesimal generator of a C_0 semigroup $T(t), t \geq 0$ on a Banach space $X, 0 \leq t_0 \leq t_1 < \ldots < t_p \leq t_0 + a, a > 0$, $u_0 \in X$ and $f : [t_0, t_0 + a] \times X \to X$, $g : [t_0, t_0 + a]^p \times X \to X$ are given functions. Balachandran and Ilamaran [1] proved the existence and uniqueness of mild and strong solutions of the problem

$$\frac{du(t)}{dt} + Au(t) = f(t, u(\sigma(t)), \quad t \in (t_0, t_0 + a]$$

$$u(t_0) + g(t_1, \ldots, t_p, u(\cdot)) = u_0.$$

Moreover, Corduneanu [3] and Gripenberg [4] studied the existence of solutions of Volterra integral equations of various types using semigroups approach.

In this paper we prove two theorems about the existence and uniqueness of the mild and strong solutions of a Volterra integrodifferential equation with nonlocal intitial condition.

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2. Preliminaries

Consider the initial value problem

$$\frac{du(t)}{dt} = Au(t) + f(t), \quad t \in (t_0, t_0 + a], \tag{1}$$

$$u(t_0) = u_0, \tag{2}$$

where $f : [t_0, t_0 + a] \to X, A$ is the infinitesimal generator of a C_0 semigroup $T(t), t \geq 0, u_0 \in X$ and $t_0 \geq 0$.

Throughout this paper we use the notation $I := [t_0, t_0 + a]$.

Difinition 1. *A function u is said to be a strong solution of problem (1), (2) on I if u is differentiable almost everywhere on I,*

$$\frac{du}{dt} \in L^1((t_0, t_0 + a], X), u(t_0) = u_0 \quad \text{and}$$

$$\frac{du(t)}{dt} = Au(t) + f(t) \quad \text{a.e. on } I$$

Theorem 1 [5]. *If X is a reflexive Banach space, $u_0 \in D(A)$ and f is Lipschitz continuous on I then problem (1), (2) has a unique strong solution u on I given by the formula*

$$u(t) = T(t - t_0)u_0 + \int_{t_0}^{t} T(t - s)f(s)ds, \quad t \in I.$$

Consider the following Volterra integrodifferential equation

$$\frac{du(t)}{dt} + Au(t) = f(t, u(t)) + \int_{t_0}^{t} K\Big(t, s, u(s), \int_{t_0}^{s} H(s, \tau, u(\tau))d\tau\Big)ds, \quad t \in (t_0, t_0 + a] \tag{3}$$

with the nonlocal condition

$$u(t_0) + g(t_1, \ldots, t_p, u(\cdot)) = u_0, \tag{4}$$

where $0 \leq t_0 < t_1 < \ldots < t_p \leq t_0 + a$, $-A$ is the infinitesimal generator of a C_0 semigroup $T(t)$, $t \geq 0$, on a Banach space X, and $f : I \times X \to X$, $g(t_1, \ldots, t_p, .) : X \to X$, $K : \Delta \times X \times X \to X$, $H : \Delta \times X \to X$ where $\Delta = \{(t, s) : t_0 \leq s \leq t \leq t_0 + a\}$. The symbol $g(t_1, \ldots, t_p, u(\cdot))$ is used in the sense that in the place of '$\cdot$' we can substitute only elements of the set $\{t_1, \ldots, t_p\}$.

Definition 2. A continuous solution u of the integral equation

$$u(t) = T(t - t_0)u_0 - T(t - t_0)g(t_1, \ldots, t_p, u(\cdot))$$
$$+ \int_{t_0}^{t} T(t-s)f(s, u(s))ds + \int_{t_0}^{t} T(t-s) \int_{t_0}^{s} K(s, \tau, u(\tau), \int_{t_0}^{\tau} H(\tau, \mu, u(\mu))d\mu)d\tau ds, \ t \in I,$$

is said to be a mild solution of problem (3), (4) on I.

3. Existence of a Mild Solution

Theorem 2. *Assume that*

(i) X *is a Banach space with norm* $\| \quad \|$ *and* $u_0 \in X$.
(ii) $0 \leq t_0 < t_1 < \ldots < t_p \leq t_0 + a$ *and* $B_r := \{v : \| v \| \leq r\} \subset X$.
(iii) $f : I \times X \to X$ *is continuous in* t *on* I *and there exists a constant* $L > 0$ *such that*

$$\| f(s, v_1) - f(s, v_2) \| \leq L \| v_1 - v_2 \| \quad \text{for } s \in I, \quad v_1, v_2 \in B_r.$$

(iv) $K : \Delta \times X \times X \to X$ *is continuous and there exists a constant* $K_0 > 0$ *such that*

$$\| K(t, s, x_1, y_1) - K(t, s, x_2, y_2) \| \leq K_0[\| x_1 - x_2 \| + \| y_1 - y_2 \|]$$

(v) $g : I^p \times X \to X$ *and there exists a constant* $G_0 > 0$ *such that*

$$\| g(t_1, \ldots, t_p, u_1(\cdot)) - g(t_1, \ldots, t_p, u_2(\cdot)) \|$$
$$\leq G_0 \sup_{t \in I} \| u_1(t) - u_2(t) \| \quad \text{for} \quad u_1, u_2 \in C(I, B_r).$$

(vi) $-A$ *is the infinitesimal generator of a* C_0 *semigroup* $T(t), t \geq 0$, *on* X
(vii) $H : \Delta \times X \to X$ *is continuous and there exists a constant* $H_0 > 0$ *such that*

$$\| H(t, s, x_1) - H(t, s, x_2) \| \leq H_0 \| x_1 - x_2 \|$$

(viii) $M = \max_{t \in [0,a]} \| T(t) \|$, $N = \max_{s \in I} \| f(s, 0) \|$,
$K_1 = \max_{t_0 \leq s \leq t \leq t_0 + a} \| K(t, s, 0, 0) \|$, $\quad H_1 = \max_{t_0 \leq s \leq t \leq t_0 + a} \| H(t, s, 0) \|$,
$G_1 = \max_{u \in C(I,B)} \| g(t_1, \ldots, t_p, u(\cdot)) \|$.
(ix) $M(\| u_0 \| + G_1 + raL + aN + K_0 ra^2 + K_0 H_0 ra^3 + K_0 H_1 a^3 + K_1 a^2) \leq r$, *and* $MG_0 + MLa + MK_0 a^2 + MK_0 H_0 a^3 < 1$.

Then problem (3), (4) *has a unique mild solution on* I.

Proof. Take $E := C(I, B_r)$ and define an operator F on E by

$$(Fv)(t) = T(t - t_0)u_0 - T(t - t_0)g(t_1, \ldots, t_p, v(\cdot)) + \int_{t_0}^{t} T(t-s)f(s, v(s))ds$$
$$+ \int_{t_0}^{t} T(t - s) \int_{t_0}^{s} K(s, \tau, v(\tau), \int_{t_0}^{\tau} H(\tau, \mu, v(\mu))d\mu)d\tau ds, t \in I,$$

From our assumption, we have

$$\| (Fv)(t) \| \leq \| T(t - t_0)u_0 \| + \| T(t - t_0)g(t_1, \ldots, t_p, v(\cdot)) \|$$

$$+\| \int_{t_0}^{t} T(t-s) f(s, v(s)) ds \| + \| \int_{t_0}^{t} T(t-s) \int_{t_0}^{s} K(s, \tau, v(\tau), \int_{t_0}^{\tau} H(\tau, \mu, v(\mu)) d\mu) d\tau ds \|$$

$$\leq M \| u_0 \| + MG_1 + M \int_{t_0}^{t} (\| f(s, v(s)) - f(s, 0) \| + \| f(s, 0) \|) ds$$

$$+ M \int_{t_0}^{t} \int_{t_0}^{s} \Big[\| K(s, \tau, v(\tau), \int_{t_0}^{\tau} H(\tau, \mu, v(\mu) d\mu) - K(s, \tau, 0, 0) \| + \| K(s, \tau, 0, 0) \| \Big] d\tau ds$$

$$\leq M \| u_0 \| + MG_1 + M \int_{t_0}^{t} (L \| v(s) \| + N) ds$$

$$+ M \int_{t_0}^{t} \int_{t_0}^{s} [K_0 \| v(\tau) \| + K_0 \int_{t_0}^{\tau} \| H(\tau, \mu, v(\mu)) \| d\mu + K_1] d\tau ds$$

$$\leq M(\| u_0 \| + G_1 + raL + aN + K_0 r a^2 + K_0 H_0 r a^3 + K_0 H_1 a^3 + K_1 a^2) \leq r, \text{ for } v \in E.$$

Therefore, $FE \subset E$.

Now, for every $v_1, v_2 \in E$ and $t \in I$, we have

$$\| (Fv_1)(t) - (Fv_2)(t) \| \leq \| T(t - t_0) \| \| g(t_1, \ldots, t_p, v_1(\cdot)) - g(t_1, \ldots, t_p, v_2(\cdot)) \|$$

$$+ \int_{t_0}^{t} \| T(t-s) \| \| f(s, v_1(s)) - f(s, v_2(s)) \| ds$$

$$+ \int_{t_0}^{t} \| T(t-s) \| \int_{t_0}^{s} \| \Big[K(s, \tau, v_1(\tau) \int_{t_0}^{\tau} H(\tau, \mu, v_2(\mu)) d\mu$$

$$- K(s, \tau, v_2(\tau), \int_{t_0}^{\tau} H(\tau, \mu, v_2(\mu)) d\mu \Big] d\tau ds$$

$$\leq MG_0 \sup_{t \in I} \| v_1(t) - v_2(t) \| + ML \int_{t_0}^{t} \| v_1(s) - v_2(s) \| ds$$

$$+ MK_0 a^2 \sup_{t \in I} \| v_1(t) - v_2(t) \| + MK_0 \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{\tau} H_0 \| v_1(\mu) - v_2(\mu) \| d\mu d\tau ds$$

$$\leq (MG_0 + MLa + MK_0 a^2 + MK_0 H_0 a^3) \sup_{t \in I} \| v_1(t) - v_2(t) \|$$

If we take $q := MG_0 + MLa + MK_0 a^2 + MK_0 H_0 a^3$ then

$$\sup_{t \in I} \| (Fv_1)(t) - (Fv_2)(t) \| \leq q \sup_{t \in I} \| v_1(t) - v_2(t) \|$$

with $0 < q < 1$.

This shows that the operator F is a contraction on the complete metric space E. By the Banach fixed point theorem the function F has a unique fixed point in the space E and this point is the mild solution of problem (3), (4) on I.

4. Existence of a Strong Solution

Definition 3. A function u is said to be a strong solution of problem (3), (4) on I if u is differentiable a.e. on I,

$$\frac{du}{dt} \in L^1((t_0, t_0 + a], X)$$
$$u(t_0) + g(t_1, \ldots, t_p, u(\cdot)) = u_0$$

and $$\frac{du(t)}{dt} + Au(t) = f(t, u(t)) + \int_{t_0}^{t} K(t, s, u(s)), \int_{t_0}^{s} H(s, \tau, u(\tau))d\tau)ds, \text{ a.e. on } I,$$

Theorem 3. *Assume that*

(i) X *is a reflexive Banach space with norm* $\| \cdot \| \delta$.

(ii) $0 \le t_0 < t_1 < \ldots < t_p \le t_0 + a$ *and* $B_r := \{v : \| v \| \le r\} \subset X$.

(iii) $f : I \times X \to X$ *is continuous in* t *on* I *and there exists a constant* $L > 0$ *such that*

$$\| f(s_1, v_1) - f(s_2, v_2) \| \le L(\| s_1 - s_2 \| + \| v_1 - v_2 \|)$$
$$\text{for} \quad s_1, s_2 \in I, v_1, v_2 \in B_r$$

(iv) $K : \Delta \times X \times X \to X$ *is continuous and there exist constant* $K_0 > 0$ *such that*

$$\| K(t_1, s, x_1, y_1) - K(t_2, s, x_2, y_2) \| \le K_0(|t_1 - t_2| + \| x_1 - x_2 \| + \| y_1 - y_2 \|)$$

(v) $H : \Delta \times X \to X$ *is continuous and there exists a constant* $G_0 > 0$ *such that*

$$\| H(t_1, s, x_1) - H(t_2, s, x_2) \| \le H_0[|t_1 - t_2| + \| x_1 - x_2 \|]$$

(vi) $g : I^p \times X \to X$ *and there exists a constant* $G_0 > 0$ *such that*

$$\| g(t_1, \ldots, t_p, u_1(\cdot)) - g(t_1, \ldots, t_p, u_2(\cdot)) \|$$
$$\le G_0 \sup_{t \in I} \| u_1(t) - u_2(t) \| \quad \text{for} \quad u_1, u_2 \in C(I, B_r), \text{ and } g(t_1, \ldots, t_p, \cdot) \in D(A).$$

(vii) $-A$ *is the infinitesimal generator of a* C_0 *semigroup* $T(t), t \ge 0$, *on* X.

(viii) $u_0 \in D(A)$.

(ix) $M = \max_{t \in [0,a]} \| T(t) \|, \quad N = \max_{s \in I} \| f(s, 0) \|,$
$K_1 = \max_{t_0 \le s \le t \le t_0 + a} \| K(t, s, 0, 0) \|, \quad H_1 = \max_{t_0 \le s \le t \le t_0 + a} \| H(t, s, 0) \|,$
$G_1 = \max_{u \in C(I, B)} \| g(t_1, \ldots, t_p, u(\cdot)) \|$.

(x) $M(\| u_0 \| + G_1 + raL + aN + K_0 ra^2 + K_0 H_0 ra^3 + K_0 H_1 a^3 + K_1 a^2) \le r$,
and $MG_0 + MLa + MK_0 a^2 + MK_0 H_0 a^3 < 1$.

Then the problem (3), (4) *has a strong solution on* I.

Proof. Since all the assumptions of Theorem 2 are satisfied then problem (3), (4) possesses a unique mild solution belonging to $C(I;X)$ which we denote by u. Now, we shall show that this mild solution is a strong solution of problem (3), (4) on I.

Take $K_2 = \max_{t_0 \le s \le t \le t_0+a} \| K(t,s,u(s),0) \|$, $H_2 = \max_{t_0 \le s \le t \le t_0+a} \| H(t,s,u(s)) \|$
$L_1 = \max_{t \in I} \| f(t,u(t)) \|$.

Then for any $t \in I$, we have

$$
\begin{aligned}
u(t+h) - u(t) &= [T(t+h-t_0) - T(t-t_0)]u_0 \\
&\quad -[T(t+h-t_0) - T(t-t_0)]g(t_1,\dots,t_p,u(\cdot)) \\
&\quad + \int_{t_0}^{t_0+h} T(t+h-s)f(s,u(s))ds + \int_{t_0+h}^{t+h} T(t+h-s)f(s,u(s))ds \\
&\quad - \int_{t_0}^{t} T(t-s)f(s,u,(s))ds + \int_{t_0}^{t_0+h} T(t+h-s)\int_{t_0}^{s} K(s,\tau,u(\tau),\int_{t_0}^{\tau} H(\tau,\mu,u(\mu))d\mu)d\tau ds \\
&\quad + \int_{t_0+h}^{t+h} T(t+h-s)\int_{t_0}^{s} K(s,\tau,u(\tau),\int_{t_0}^{\tau} H(\tau,\mu,u(\mu))d\mu)d\tau ds \\
&\quad - \int_{t_0}^{t} T(t-s)\int_{t_0}^{s} K(s,\tau,u(\tau),\int_{t_0}^{\tau} H(\tau,\mu,u(\mu))d\mu)d\tau ds \\
&= [T(t+h-t_0) - T(t-t_0)]u_0 - [T(t+h-t_0) - T(t-t_0)]g(t_1,\dots,t_p,u(\cdot)) \\
&\quad + \int_{t_0}^{t_0+h} T(t+h-s)f(s,u(s))ds \\
&\quad + \int_{t_0}^{t} T(t-s)[f(s+h,u(s+h)) - f(s,u(s))]ds \\
&\quad + \int_{t_0}^{t_0+h} T(t+h-s)\int_{t_0}^{s} K(s,\tau,u(\tau),\int_{t_0}^{\tau} H(\tau,\mu,u(\mu))d\mu)d\tau ds \\
&\quad + \int_{t_0}^{t} T(t-s)\int_{t_0}^{s+h} K(s+h,\tau,u(\tau),\int_{t_0}^{\tau} H(\tau,\mu,u(\mu))d\mu)d\tau ds \\
&\quad - \int_{t_0}^{t} T(t-s)\int_{t_0}^{s} K(s,\tau,u(\tau),\int_{t_0}^{\tau} H(\tau,\mu,u(\mu))d\mu)d\tau ds \\
&= T(t-t_0)[T(h)-I]u_0 - T(t-t_0)[T(h)-I]g(t_1,\dots,t_p,u(\cdot))] \\
&\quad + \int_{t_0}^{t_0+h} T(t+h-s)f(s,u(s))ds \\
&\quad + \int_{t_0}^{t} T(t-s)[f(s+h,u(s+h)) - f(s,u(s))]ds \\
&\quad + \int_{t_0}^{t_0+h} T(t+h-s)\int_{t_0}^{s} [K(s,\tau,u(\tau),\int_{t_0}^{\tau} H(\tau,\mu,u(\mu))d\mu) - K(s,\tau,u(\tau),0)]d\tau ds
\end{aligned}
$$

$$+\int_{t_0}^{t_0+h} T(t+h-s)\int_{t_0}^{s} K(s,\tau,u(\tau),0)d\tau ds$$
$$+\int_{t_0}^{t} T(t-s)\int_{t_0}^{s}[K(s+h,\tau,u(\tau),\int_{t_0}^{\tau} H(\tau,\mu,u(\mu))d\mu)$$
$$-K(s,\tau,u(\tau),\int_{t_0}^{\tau} H(\tau,\mu,u(\mu))d\mu)]d\tau ds$$
$$+\int_{t_0}^{t} T(t-s)\int_{s}^{s+h}[K(s+h,\tau,u(\tau),\int_{t_0}^{\tau} H(\tau,\mu,u(\mu))d\mu)-K(s+h,\tau,u(\tau),0)]d\tau ds$$
$$+\int_{t_0}^{t} T(t-s)\int_{s}^{s+h} K(s+h,\tau,u(\tau),0)d\tau ds$$

Using our assumptions we observe that

$$\begin{aligned}\| u(t+h)-u(t)\| &\le hM\| Au_0\| +hM\| Ag(t_1,\dots,t_p,u(\cdot))\| +hML_1\\ &\quad +MLah+ML\int_{t_0}^{t}\| u(s+h)-u(s)\| ds\\ &\quad + MK_0H_2a^2h+MK_2ah+MK_0a^2h+MK_0H_2a^2h+MK_2ah\\ &\le Ph+ML\int_{t_0}^{t}\| u(s+h)-u(s)\| ds,\end{aligned}$$

where $P := M\| Au_0\| + M\| Ag(t_1,\dots,t_p,u(\cdot))\| +ML_1+MLa + MK_0H_2a^2+MK_2a+MK_0a^2+MK_0H_2a^2+MK_2a.$

Using Gronwall's inequality, we get

$$\| u(t+h)-u(t)\| \le Phe^{MLa} \quad \text{for} \quad t\in I.$$

Therefore, u is Lipschitz continuous on I.

The Lipschitz continuity of u on I combined with (iii) give that $t\to f(t,u(t))$ is Lipschitz continuous on I. Also, by assumption (iv) and (v) $t\to \int_{t_0}^{t} K(t,s,u(s),$ $\int_{t_0}^{s} H(s,\tau,u(\tau))d\tau)ds$ is Lipschitz continuous on I. Using Theorem 1 we observe that the equation

$$\frac{dv(t)}{dt}+Av(t)=f(t,u(t))+\int_{t_0}^{t} K(t,s,u(s),\int_{t_0}^{s} H(s,\tau,u(\tau))d\tau)ds, \quad t\in (t_0,t_0+a]$$
$$v(t_0)=u_0-g(t_1,\dots,t_p,u(\cdot))$$

has a unique strong solution v on I satisfying the equation

$$\begin{aligned}v(t)&=T(t-t_0)u_0-T(t-t_0)g(t_1,\dots,t_p,u(\cdot))\\ &\quad +\int_{t_0}^{t} T(t-s)f(s,u(s))ds+\int_{t_0}^{t} T(t-s)\int_{t_0}^{s} K(s,\tau,u(\tau),\int_{t_0}^{\tau} H(\tau,\mu,u(\mu)d\mu)d\tau ds\\ &= u(t), \quad t\in I.\end{aligned}$$

Consequently, u is a strong solution of problem (3), (4) on I.

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