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ON THE EXTREMAL CURVATURE AND TORSION OF STEREOGRAPHICALLY PROJECTED ANALYTIC CURVES

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Dedicated to William E. Kirwan, in gratitude

Abstract. In this paper, we first derive formulas for the curvature and torsion of curves on S^2 produced by stereographically projecting the image curves of analytic, univalent functions belonging to the class $\mathcal{S}$. We are concerned here with the problems of determining the extreme values of the curvatures and torsions, as well as the functions belonging to $\mathcal{S}$ which attain these extreme values. An analysis of the asymptotic behavior of these curvature and torsion formulas will allow for the formulation of plausible conjectures.

1. Introduction

Let $\mathcal{S}$ denote the class of analytic, univalent functions $f(z)$ defined on the unit disk $\mathcal{U} = \{z : |z| < 1\}$, and normalized so that $f(0) = 0$ and $f'(0) = 1$.

Let $f \in \mathcal{S}$ and let Π denote the stereographic projection of the image plane of f onto the unit sphere S^2. For each fixed r, $0 < r < 1$, let $\mathcal{C}_r = \{z : |z| = r\}$, $\mathcal{C}'_r = f(\mathcal{C}_r)$, and $\mathcal{C}''_r = \Pi(\mathcal{C}'_r)$. For each fixed θ, $-\pi < \theta \leq +\pi$, let $\mathcal{L}_\theta = \{z : arg\, z = \theta\}$, $\mathcal{L}'_\theta = f(\mathcal{L}_\theta)$, and $\mathcal{L}''_\theta = \Pi(\mathcal{L}'_\theta)$.

The determination of the extreme values of the local curvature at a specified point on each of the curves $\mathcal{C}'_r$ and $\mathcal{L}'_\theta$ has been the object of an intense body of research [1, pp.126, 262]. Of course, the local torsion at any specified point on each of these curves is equal to zero, since they are plane curves.

Our ultimate objective is to maximize and minimize the local curvature and torsion at a specified point on the stereographically projected curves $\mathcal{C}''_r$ and $\mathcal{L}''_\theta$ on the unit sphere S^2. Note that the stereographically induced torsion will generally be a non-zero quantity. It is this fact which motivates our investigation.

A parametrization of the curves under consideration is easily prescribed. Indeed, if we write

$$f(z) = f(r,\theta) = (u(r,\theta), v(r,\theta)), \quad (z = re^{i\theta} \in \mathcal{U}) \tag{1.1}$$

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then

$$\mathcal{C}''_r = \{X(r,\theta) : -\pi < \theta \leq +\pi\},$$

and

$$\mathcal{L}''_\theta = \{X(r,\theta) : 0 < r < 1\},$$

where

$$X(r,\theta) = \Big(\frac{2u(r,\theta)}{|f|^2+1}, \frac{2v(r,\theta)}{|f|^2+1}, \frac{|f|^2-1}{|f|^2+1}\Big). \tag{1.2}$$

In terms of the parametrization (1.2), the local curvature $\kappa(\theta;r,f)$ and the local torsion $\tau(\theta;r,f)$ at the point $X(r,\theta)$ on the curve $\mathcal{C}''_r$ are classically defined by the formulas

$$\kappa(\theta;r,f) = \frac{|X_\theta \times X_{\theta\theta}|}{|X_\theta|^3} \quad \text{and} \quad \tau(\theta;r,f) = \frac{[X_\theta X_{\theta\theta} X_{\theta\theta\theta}]}{|X_\theta \times X_{\theta\theta}|^2}.$$

The local curvature $\kappa(r;\theta,f)$ and torsion $\tau(r;\theta,f)$ at the point $X(r,\theta)$ on the curve $\mathcal{L}''_\theta$ are given by similar formulas. The subscripts on X denote the variable with respect to which the derivative is taken.

In our first result below, we provide explicit formulas for the curvatures and torsions under study. In this and all results to follow, the quantity

$$f^{\#}(z) = \frac{|f'(z)|}{1+|f(z)|^2} \tag{1.3}$$

will denote the *spherical derivative of* $f(z)$, and the quantity

$$\{f,z\} = \Big(\frac{f''(z)}{f'(z)}\Big)' - \frac{1}{2}\Big(\frac{f''(z)}{f'(z)}\Big)^2 \tag{1.4}$$

will denote the *Schwarzian derivative of* $f(z)$. Note that equation (1.7) below provides and explicit connection between these two quantities.

Theorem 1.1. *Let* $f \in \mathcal{S}$, *and let* $\mathcal{Z}(r,\theta;f) = \frac{1}{rf^{\#}(r,\theta)}$.

(a) At the point $X(r,\theta)$ *on the curve* $\mathcal{C}''_r$, *the local curvature* $\kappa(\theta;r,f)$ *is given by the formula*

$$\kappa(\theta;r,f) = \Big(1 + \frac{1}{4}r^2\mathcal{Z}_r^2(r,\theta;f)\Big)^{\frac{1}{2}}, \tag{1.5}$$

and the local torsion $\tau(\theta;r,f)$ *is given by the formula*

$$\tau(\theta;r,f) = \frac{r\mathcal{Z}(r,\theta;f)\mathcal{Z}_{r\theta}(r,\theta;f)}{4\kappa^2(\theta;r,f)}. \tag{1.6}$$

Furthermore, since

$$\frac{r\mathcal{Z}_{r\theta}(r,\theta;f)}{\mathcal{Z}(r,\theta;f)} = \operatorname{Im}[z^2\{f,z\}], \tag{1.7}$$

all of these quantities are related by

$$4r^2 f^{\#2}(r,\theta)\kappa^2(\theta;r,f)\tau(\theta;r,f) = \mathrm{Im}[z^2\{f,z\}]. \tag{1.8}$$

(*b*) *At the point* $X(r,\theta)$ *on the curve* $\mathcal{L}''_\theta$, *the local curvature* $\kappa(r;\theta,f)$ *is given by the formula*

$$\kappa(r;\theta,f) = \left(1+\frac{1}{4}\mathcal{Z}_\theta^2(r,\theta;f)\right)^{\frac{1}{2}}, \tag{1.9}$$

and the local torsion $\tau(r;\theta,f)$ *is given by the formula*

$$\tau(r;\theta,f) = -\frac{r\mathcal{Z}(r,\theta;f)\mathcal{Z}_{r\theta}(r,\theta;f)}{4\kappa^2(r;\theta,f)}. \tag{1.10}$$

Furthermore, all of these quantities are related by

$$4r^2 f^{\#2}(r,\theta)\kappa^2(r;\theta,f)\tau(r;\theta,f) = -\mathrm{Im}[z^2\{f,z\}]. \tag{1.11}$$

The proof of this result is lengthy, but straightforward, and will be omitted.

In deriving formulas (1.5) and (1.9), it becomes clear that the partial derivatives $-r\mathcal{Z}_r(r,\theta;f)$ and $\mathcal{Z}_\theta(r,\theta;f)$ are actually the real and imaginary parts of the same quantity. Indeed, if we set

$$\Phi(z;f) = \frac{\left(1+\frac{zf''(z)}{f'(z)}\right)-\left(\frac{2|f(z)|^2}{1+|f(z)|^2}\right)\left(\frac{zf'(z)}{f(z)}\right)}{2rf^{\#}(z)} \tag{1.12}$$

then

$$\mathrm{Re}\{\Phi(z;f)\} = -\frac{1}{2}r\mathcal{Z}_r(r,\theta;f), \tag{1.13}$$

and

$$\mathrm{Im}\{\Phi(z;f)\} = +\frac{1}{2}\mathcal{Z}_\theta(r,\theta;f). \tag{1.14}$$

Hence, the curvature functions are related to each other via $\Phi(z;f)$.

Some consequences of this theorem are easily noted. For example, an addition of (1.8) and (1.11) yields the *complementary relation*

$$\kappa^2(\theta;r,f)\tau(\theta;r,f)+\kappa^2(r;\theta,f)\tau(r;\theta,f) = 0,$$

which is valid at the point $X(r,\theta)\in S^2$ of intersection of the two curves $\mathcal{C}''_r$ and $\mathcal{L}''_\theta$. It may be deduced from this relation that either both torsions are zero at such a point or that they are of opposite sign.

A consequence of (1.6), (1.7), and (1.10) is that both torsions are zero at a point $X(r,\theta)\in S^2$ if and only if $\mathrm{Im}[z^2\{f,z\}]=0$ at the corresponding point $z=re^{i\theta}\in\mathcal{U}$. Thus, the level lines of the harmonic function $\mathrm{Im}[z^2\{f,z\}]$ as well as its sign will be of significance in our study of the torsion.

At this point, we present some examples in order to familiarize the reader with the algebraic form of the curvature and torsion functions. The examples presented here will be of significance in the later sections of this paper.

Example 1. *Linear-Fractional Transformations.* Let $f(z) = z/(1 - cz)$, where $c \in [0, 1]$. Since f is linear-fractional, the stereographically projected images $\mathcal{C}''_r$ and $\mathcal{L}''_\theta$ of the circular of linear arcs $\mathcal{C}'_r$ and $\mathcal{L}'_\theta$ are circular arcs. Consequently, $\tau(r; \theta, f) = \tau(\theta; r, f) = 0$ for every value of r, θ, and c.

A brief calculation shows that

$$\kappa(\theta; r, f) = \frac{1}{2r}\sqrt{1 + 2(1 - c^2)r^2 + (1 + c^2)^2 r^4},$$

showing that, as expected, the curvature of the circle $\mathcal{C}''_r$ is independent of θ.

Another brief calculation shows that

$$\kappa(r; \theta, f) = \sqrt{1 + c^2 \sin^2 \theta}$$

showing that, as expected, the curvature of the circular arc L''_θ is independent of r.

Example 2. *Some Well-Known Slit Mappings.* Let

$$f(z) = \frac{z}{1 - 2tz + z^2}. \quad (t \in [0, 1]) \tag{1.15}$$

We shall suppress the dependence upon t in order to minimize notation. This function belongs to the class $\mathcal{S}$ for each value of $t \in [0, 1]$. If $t \in [0, 1)$, it maps the unit disk U onto the complement of two linear, opposing, radially directed slits. If $t = 1$, this function becomes the well-known Koebe function, which maps U onto the complement of the single linear interval $(-\infty, -\frac{1}{4}]$.

For the function (1.15), we have

$$\mathcal{Z}(r, \theta; f) = \frac{1 - (4t\cos\theta)r + (1 + 4t^2 + 2\cos 2\theta)r^2 - (4t\cos\theta)r^3 + r^4}{r\sqrt{1 - 2r^2\cos 2\theta + r^4}}. \tag{1.16}$$

Note the symmetry in this formula and the formulas to follow in this example.

Evaluating the first partials of $\mathcal{Z}(r, \theta; f)$, we find that

$$-\frac{1}{2} r \mathcal{Z}_r(r, \theta; f) = \frac{(1 - r^2)\left(\sum\limits_{i=0}^{6} a_i(\theta) r^i\right)}{2r(1 - 2r^2\cos 2\theta + r^4)^{3/2}}, \tag{1.17}$$

where $a_0(\theta) = a_6(\theta) = 1$,
$a_1(\theta) = a_5(\theta) = 0$,
$a_2(\theta) = a_4(\theta) = -(4t^2 + 6\cos 2\theta)$,
and
$a_3(\theta) = 4t(3\cos\theta + \cos 3\theta)$,

and that

$$\frac{1}{2}\mathcal{Z}_\theta(r,\theta;f) = \frac{2(\sin\theta)\left(\sum_{i=0}^{6} b_i(\theta)r^i\right)}{(1-2r^2\cos 2\theta + r^4)^{3/2}}, \tag{1.18}$$

where $b_0(\theta) = b_6(\theta) = t$,
$b_1(\theta) = b_5(\theta) = -3\cos\theta$,
$b_2(\theta) = b_4(\theta) = 3t$,
and
$b_3(\theta) = -4t^2\cos\theta + \cos 3\theta$.

Consequently, the curvature $\kappa(\theta;r,f)$ at the point $X(r,\theta)$ on the curve C_r'' is given by

$$\kappa(\theta;r,f) = \frac{1}{2r}\sqrt{\frac{\sum_{i=0}^{16} c_i(\theta)r^i}{(1-2r^2\cos 2\theta + r^4)^3}}, \tag{1.19}$$

where $c_0(\theta) = c_{16}(\theta) = 1$,
$c_1(\theta) = c_{15}(\theta) = 0$,
$c_2(\theta) = c_{14}(\theta) = 2 - 8t^2 - 12\cos 2\theta$,
$c_3(\theta) = c_{13}(\theta) = 8t(3\cos\theta + \cos 3\theta)$,
$c_4(\theta) = c_{12}(\theta) = 19 + 8t^2 + 16t^4 + 12(4t^2-1)\cos 2\theta + 18\cos 4\theta$,
$c_5(\theta) = c_{11}(\theta) = -48t(3+t^2)\cos\theta - 8t(11+4t^2)\cos 3\theta - 24t\cos 5\theta$,
$c_6(\theta) = c_{10}(\theta) = 38 + 88t^2 + 12(1+10t^2)\cos 2\theta + 24(1+2t^2)\cos 4\theta$,
$c_7(\theta) = c_9(\theta) = 24t(5+4t^2)\cos\theta + 16t(5+2t^2)\cos 3\theta + 24t\cos 5\theta$.
and
$c_8(\theta) = -8(5+22t^2+4t^4) - 48(2+7t^2)\cos 2\theta - 12(3+8t^2)\cos 4\theta - 8(1+2t^2)\cos 6\theta$.

In a similar manner, it can be seen that the curvature $\kappa(r;\theta,f)$ at the point $X(r,\theta)$ on the curve L_θ'' is given by

$$\kappa(r;\theta,f) = \sqrt{\frac{\sum_{i=0}^{12} d_i(\theta)r^i}{2(1-2r^2\cos 2\theta + r^4)^3}}, \tag{1.20}$$

where $d_0(\theta) = d_{12}(\theta) = 2 + 4t^2 - 4t^2\cos 2\theta$,
$d_1(\theta) = d_{11}(\theta) = -12t(\cos\theta + \cos 3\theta)$,
$d_2(\theta) = d_{10}(\theta) = 3(3+8t^2) - 12(1+2t^2)\cos 2\theta - 9\cos 4\theta$,
$d_3(\theta) = d_9(\theta) = -8t(5+2t^2)\cos\theta + 4t(11+4t^2)\cos 3\theta - 4t\cos 5\theta$,
$d_4(\theta) = d_8(\theta) = 4(6+21t^2) - 6(1+10t^2)\cos 2\theta + 6(1-4t^2)\cos 4\theta + 6\cos 6\theta$,
$d_5(\theta) = d_7(\theta) = -12t(5+4t^2)\cos\theta + 24t(3+2t^2)\cos 3\theta - 12t\cos 5\theta$,
and
$d_6(\theta) = 4(5+22t^2+4t^4) - 2(19+44t^2)\cos 2\theta - (19+8t^2+16t^4)\cos 4\theta$
$-2(1-8t^2)\cos 6\theta - \cos 8\theta$.

An additional calculation shows that

$$r\mathcal{Z}_{r\theta}(r,\theta;f) = \frac{6r^2(r^4-1)(\sin 2\theta)\mathcal{Z}(r,\theta;f)}{(1-2r^2\cos 2\theta + r^4)^2}. \tag{1.21}$$

Hence, the torsion $\tau(\theta; r, f)$ at the point $X(r,\theta)$ on the curve C_r'' is given by

$$\tau(\theta; r, f) = \frac{6r^4(r^4-1)(1-2r^2\cos 2\theta + r^4)(\sin 2\theta)\mathcal{Z}^2(r,\theta; f)}{\left(\sum_{i=0}^{16} c_i(\theta) r^i\right)}$$

and the torsion $\tau(r; \theta, f)$ at the point $X(r,\theta)$ on the curve L_θ'' is given by

$$\tau(r; \theta, f) = -\frac{3}{2}\frac{r^2(r^4-1)(1-2r^2\cos 2\theta + r^4)(\sin 2\theta)\mathcal{Z}^2(r,\theta; f)}{\left(\frac{1}{2}\sum_{i=0}^{12} d_i(\theta) r^i\right)}.$$

In §2, a discussion of the extremal problems under consideration is presented, and a method is described for determining the extreme values of the curvature functions of a single function $f \in \mathcal{S}$. In §3, asymptotic properties of the curvature and torsion functions for small values of r are explored, yielding some plausible conjectures. Finally, in §4, some open problems relating to curvature and torsion are posed.

2. A Discussion of The Extremal Problems

Initially, for fixed values of r and θ, we propose to determine the eight extremal values

$$K(r) = \max_{f\in\mathcal{S}} \kappa(\theta; r, f) \quad k(r) = \min_{f\in\mathcal{S}} \kappa(\theta; r, f)$$

$$K(\theta) = \max_{f\in\mathcal{S}} \kappa(r; \theta, f) \quad k(\theta) = \min_{f\in\mathcal{S}} \kappa(r; \theta, f)$$

$$T(r) = \max_{f\in\mathcal{S}} \tau(\theta; r, f) \quad t(r) = \min_{f\in\mathcal{S}} \tau(\theta; r, f)$$

$$T(\theta) = \max_{f\in\mathcal{S}} \tau(r; \theta, f) \quad t(\theta) = \max_{f\in\mathcal{S}} \tau(r; \theta, f)$$

and the extremal functions for which these extremal values are attained.

Without loss of generality, we may assume that $\theta = 0$ in these eight problems. To see this, we first note some elementary rotational properties of the curvature and torsion functions.

If we set $f_\epsilon(z) = e^{-i\epsilon} f(e^{i\epsilon} z)$, then the range of $f(z)$ is rotated by a factor of ϵ, causing a corresponding shift in the θ-variable of the quantities in question. Specifically, brief calculations show that

$$\kappa(\theta; r, f_\epsilon) = \kappa(\theta + \epsilon; r, f),$$

$$\tau(\theta; r, f_\epsilon) = \tau(\theta + \epsilon; r, f),$$

$$\kappa(r; \theta, f_\epsilon) = \kappa(r; \theta + \epsilon, f),$$

and

$$\kappa(r; \theta, f_\epsilon) = \tau(r; \theta + \epsilon, f).$$

Thus, $\kappa(0; r, f) = \kappa(\theta; r, f_{-\theta})$, showing that the extreme value of this curvature function when $\theta = 0$ is the same as the extreme value for an arbitrary θ, and that the extremal functions are related by a simple rotation.

Some additional geometric properties of the curvature and torsion are worth noting here.

If we set $\overline{f} = \overline{f(\overline{z})}$, then the range of $f(z)$ is reflected over the real axis, causing a corresponding sign change in the θ-variable of the quantities in question. Specifically, another set of brief calculations shows that

$$\kappa(\theta; r, \overline{f}) = \kappa(-\theta; r, f),$$

$$\tau(\theta; r, \overline{f}) = -\tau(-\theta; r, f),$$

$$\kappa(r; \theta, \overline{f}) = \kappa(r; -\theta, f),$$

and

$$\tau(r; \theta, \overline{f}) = -\tau(r; -\theta, f).$$

Although it appears on the surface that there are eight extremal problems to solve, there are actually fewer. Indeed, the problems of determining the extreme values of the curvatures as defined by (1.5) and (1.9) are equivalent to the problems of determining the extreme values of $|r\mathcal{Z}_r|$ and $|\mathcal{Z}_\theta|$. In view of (1.13) and (1.14), all of the curvature problems are aspect of *one* problem, namely,

$$\max_{f \in \mathcal{S}} \operatorname{Re}\{\eta \Phi(z; f)\} \tag{2.1}$$

where $|\eta| = 1$, and $\Phi(z; f)$ is defined by (1.12). One only need to choose $\eta = \pm 1$ or $\pm i$ to see this.

An elementary observation shows that one of the minimal local curvature problems is trivial.

Theorem 2.1. *For every $\theta \in (-\pi, +\pi]$, and every $r \in (0, 1)$, we have*

$$k(\theta) = \min_{f \in \mathcal{S}} \kappa(r; \theta, f) \equiv 1.$$

Proof. If $f(z) = z$, then $f^{\#}(r, \theta) = 1/(1 + r^2)$. It follows that $\mathcal{Z}_\theta(r, \theta; f) \equiv 0$, so that $\kappa(r; \theta, f) \equiv 1$. There are actually an infinite number of functions for which this minimum is attained.

A partial result is also easily established for the other minimal local curvature problem.

Theorem 2.2. *for every $r \in [\widehat{r}, 1)$ and every $\theta \in (-\pi, +\pi]$, we have*

$$k(r) = \min_{f \in \mathcal{S}} \kappa(\theta; r, f) = 1,$$

where $\widehat{r} = 0.2597148103\ldots$ *is the only root of the equation*

$$1 - 10r^2 - 16r^3 - 10r^4 + r^6 = 0$$

in the interval $(0, 1)$.

Proof. It is clear from (1.5) that $\kappa(\theta; r, f) = 1$ if and only if $\mathcal{Z}_r(r, \theta; f) = 0$. For the functions defined by (1.15), we will have $\mathcal{Z}_r(r, \theta; f) = 0$ if and only if

$$\begin{aligned} A(r, \theta, t) &= \sum_{i=0}^{6} a_i(\theta) r^i \\ &= 1 - (4t^2 + 6\cos 2\theta) r^2 + 4t(3\cos\theta + \cos 3\theta) r^3 - (4t^2 + 6\cos 2\theta) r^4 + r^6 \end{aligned}$$

determined from (1.17) vanishes for some choice of the variables r, θ, and t. If we choose $\theta = \pi$ and $t = 1$, then

$$A(r, \pi, 1) = 1 - 10r^2 - 16r^3 - 10r^4 + r^6 = 0$$

if $r = \widehat{r} = 0.2597148103\cdots$. Furthermore, $A(r, \pi, 1) < 0$ on the interval $(\widehat{r}, 1)$, and $A(r, \frac{\pi}{2}, 0) = 1 + 6r^2 + 6r^4 + r^6 > 0$ for each $r \in (0, 1)$. Hence, for each fixed $\rho \in (\widehat{r}, 1)$, we may apply the Intermediate Value Theorem to the continuous function $A(\rho, \theta, t)$ restricted to the cross-sectional rectangle $R_\rho = \{(\rho, \theta, t) : 0 \le \theta \le 2\pi, 0 \le t \le 1\}$ to conclude that there must exist values θ_ρ and t_ρ for which $A(\rho, \theta_\rho, t_\rho) = 0$. That is, for each $\rho \in (\widehat{r}, 1)$, there exists θ_ρ and t_ρ for which $\kappa(\theta_\rho; \rho, f) = 1$, which was to be shown. There are actually an infinite number of functions for which this minimum is attained.

For any given $f \in \mathcal{S}$, it is easy to locate the critical points of the curvature $\kappa(\theta; r, f)$ on C''_r and the critical points of the curvature $\kappa(r; \theta, f)$ on $\mathcal{L}''_\theta$. By differentiating (1.5), we obtain

$$\kappa_\theta(\theta; r, f) = \frac{r^2 \mathcal{Z}_r(r, \theta; f) \mathcal{Z}_{r\theta}(r, \theta; f)}{4\kappa(\theta; r, f)}.$$

Thus, for a critical point of the curvature $\kappa(\theta; r, f)$ to occur at a point $X(r, \theta) \in C''_r$ it is necessary that either $\mathcal{Z}_r(r, \theta; f) = 0$ or $\mathcal{Z}_{r\theta}(r, \theta; f) = 0$. If $\mathcal{Z}_r(r, \theta; f) = 0$, then $\kappa(\theta; r, f) = 1$, a minimum value. If $\mathcal{Z}_{r\theta}(r, \theta; f) = 0$, then, by (1.6) and (1.7), it must be the case that the torsion $\tau(\theta; r, f) = 0$ and $\mathrm{Im}[z^2\{f, z\}] = 0$, where $z = re^{i\theta} \in \mathcal{U}$. Consequently, if a local maximum value of the curvature $\kappa(\theta; r, f)$ occurs at a point $X(r, \theta) \in C''_r$, then the torsion vanishes at that point and $\mathrm{Im}[z^2\{f, z\}] = 0$ at the corresponding point $z = re^{i\theta} \in \mathcal{U}$. Also, by differentiating (1.9), we obtain

$$\kappa_r(r; \theta, f) = \frac{\mathcal{Z}_\theta(r, \theta; f) \mathcal{Z}_{r\theta}(r, \theta; f)}{4\kappa(r; \theta, f)}.$$

Thus, for a critical point of the curvature $\kappa(r; \theta, f)$ to occur at a point $X(r, \theta) \in \mathcal{L}''_\theta$, it is necessary that either $\mathcal{Z}_\theta(r, \theta; f) = 0$ or $\mathcal{Z}_{r\theta}(r, \theta; f) = 0$. If $\mathcal{Z}_\theta(r, \theta; f) = 0$, then $\kappa(r; \theta, f) = 1$, a minimum value. If $\mathcal{Z}_{r\theta}(r, \theta; f) = 0$, then the torsion $\tau(r; \theta, f) = 0$

and $\text{Im}[z^2\{f,z\}] = 0$, where $z = re^{i\theta} \in \mathcal{U}$. Consequently, if a maximum value of the curvature $\kappa(r;\theta,f)$ occurs at a point $X(r,\theta) \in \mathcal{L}''_\theta$, then the torsion vanishes at that point and $\text{Im}[z^2\{f,z\}] = 0$ at the corresponding point $z = re^{i\theta} \in \mathcal{U}$. Procedurally then, to determine the local maximum values of either of the curvatures, we first determine the set $\Gamma_0 = \{z : \text{Im}[z^2\{f,z\}] = 0\}$ of level lines in $\mathcal{U}$, and then examine the values of each of the curvature functions on the sets $\Gamma_0 \cap \mathcal{C}_r \subset \mathcal{U}$ and $\Gamma_0 \cap \mathcal{L}_\theta \subset \mathcal{U}$ of critical points.

Additionally, we need only solve the two maximal local torsion problems. The two minimal local torsion problems are easily seen to be completely equivalent to them, since $\tau(0;r,\overline{f}) = -\tau(0;r,f)$, and $\tau(r;0,\overline{f}) = -\tau(r;0,f)$.

3. Asymptotic Estimates for Small Values of r

Insight into the extreme behavior of the curvature and torsion of stereographically projected curves may be gained by an examination of their asymptotic behaviors for small values of r. In particular, the formulation of conjectures concerning the extreme values of the curvatures and torsions and the functions which attain those values is possible as a result of such an examination. Furthermore, the elimination of candidates for extremal functions is also possible. Toward these ends, we compute the series expansions of the curvature and torsion functions about the origin. However, in order to formulate conjectures properly, we must first recall some basic facts from the elementary theory of univalent functions.

It is well-known that the *Koebe function*

$$\kappa_\alpha(z) = \frac{z}{(1-e^{i\alpha}z)^2} = z + 2e^{i\alpha}z^2 + 3e^{2i\alpha}z^3 + \cdots \tag{3.1}$$

belongs to the class $\mathcal{S}$ and is the solution to many linear and nonlinear extremal problems posed for that class, as is its it square root transform

$$h_{\alpha/2}(z) = \sqrt{k_\alpha(z^2)} = \frac{z}{1-(e^{i\alpha/2}z)^2}.$$

These functions will also assume importance here.

Let $\sum$ denote the class of functions $g(z)$ which are meromorphic and univalent on the set $\Delta = \{z : |z| > 1\}$ and are normalized at ∞ so that $g(\infty) = \infty$ and $g'(\infty) = 1$. Any function $g(z) \in \sum$ will have an expansion of the form

$$g(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots$$

The Area Theorem [2, p.29] asserts that the coefficients of the functions in $\sum$ must satisfy the inequality

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1.$$

Now, if $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{S}$, then $g(z) = 1/f(1/z) \in \sum$, and has the representation

$$g(z) = z - a_2 + \frac{(a_2^2 - a_3)}{z} - \frac{a_4 - 2a_2 a_3 + a_2^3}{z^2} + \cdots$$

Thus, if $g(z)$ is produced from a function $f \in \mathcal{S}$ in this manner, and $|b_1| = 1$, then the Area Theorem implies that $b_n = 0$ for all $n \geq 2$, so that the series for $g(z)$ terminates, $g(z)$ assumes the form

$$g(z) = z - a_2 + \frac{(a_2^2 - a_3)}{z}, \tag{3.2}$$

and $f(z)$ assumes the corresponding form

$$f(z) = \frac{z}{1 - a_2 z + (a_2^2 - a_3)z^2}. \tag{3.3}$$

Since a function $g(z)$ of the form (3.2) maps Δ onto the exterior of a linear segment which contains the origin, the corresponding function $f(z)$ will map $\mathcal{U}$ onto the complement of a radially directed linear segment which contains the point at infinity. The mapping properties, as well as the curvatures and torsions of such a function, were discussed in Example 2 of §1. Since $f(z)$ has no poles in $\mathcal{U}$, both roots of the denominator must have absolute value equal to one. Hence, f may be rewritten in the form

$$f(z) = \frac{z}{(1 - \overline{\xi} z)(1 - \overline{\eta} z)}, \quad (|\eta| = |\xi| = 1) \tag{3.4}$$

where $a_2 = \overline{\xi} + \overline{\eta}$ and $a_2^2 - a_3 = \overline{\xi}\,\overline{\eta}$. Conversely, if a function $f(z)$ has the form (3.4), then $a_2 = \overline{\xi} + \overline{\eta}$, $a_3 = \overline{\xi}^2 + \overline{\xi}\,\overline{\eta} + \overline{\eta}^2$, and $a_4 = \overline{\xi}^3 + \overline{\xi}^2\overline{\eta} + \overline{\xi}\,\overline{\eta}^2 + \overline{\eta}^3$.

With these preliminary comments out of the way, we can now return to the asymptotic nature of the geometric quantities under discussion.

For simplicity in notation, let us write

$$r f^{\#}(r, \theta) = \sum_{n=0}^{\infty} d_n(\theta) r^{n+1},$$

and

$$\mathcal{Z}(r, \theta; f) = \frac{1}{r f^{\#}(r, \theta)} = \frac{1}{r} + \sum_{n=0}^{\infty} e_n(\theta) r^n, \tag{3.5}$$

where $d_n(\theta)$ and $e_n(\theta)$ are infinitely differentiable, 2π-periodic functions of θ. The first few coefficients in these series are given by

$d_0(\theta) = 1$,

$d_1(\theta) = 2\mathrm{Re}\{a_2 e^{i\theta}\}$,

$d_2(\theta) = -1 + 3\mathrm{Re}\{a_3 e^{2i\theta}\} + 2(\mathrm{Im}\{a_2 e^{i\theta}\})^2$,

and

$e_0(\theta) = -2\mathrm{Re}\{a_2 e^{i\theta}\}$,

$e_1(\theta) = 1 + |a_2|^2 + 3\mathrm{Re}\{(a_2^2 - a_3)e^{2i\theta}\}$,

$e_2(\theta) = -2\text{Re}\{2a_4e^{3i\theta} + 3a_3\bar{a}_2e^{i\theta}\} + 6\text{Re}\{a_2e^{i\theta}\}\text{Re}\{a_2^2e^{2i\theta}\} + 4(\text{Re}\{a_2e^{i\theta}\})^3 -$ $18\text{Re}\{a_2e^{i\theta}\}\text{Re}\{(a_2^2 - a_3)e^{2i\theta}\}$.

The Curvature $\kappa(\theta; r, f)$.

An elementary calculation shows that

$$\mathcal{Z}_r(r, \theta; f) = -\frac{1}{r^2} + \sum_{n=1}^{\infty} ne_n(\theta)r^{n-1},$$

and that

$$\mathcal{Z}_r^2(r, \theta; f) = \frac{1}{r^4} - \frac{2e_1(\theta)}{r^2} - \frac{4e_2(\theta)}{r} + (e_1^2(\theta) - 6e_3(\theta)) + \sum_{n=1}^{\infty} E_n(\theta)r^n,$$

where

$$E_n(\theta) = \sum_{j=1}^{n+1} j(n+2-j)e_j(\theta)e_{n+2-j}(\theta) - 2(n+3)e_{n+3}(\theta). \quad (n \geq 1)$$

Hence, from (1.5), we obtain

$$\kappa(\theta; r, f) = \frac{1}{2r} + \frac{1}{2}(1 - |a_2|^2 - 3\text{Re}\{(a_2^2 - a_3)e^{2i\theta}\})r - e_2(\theta)r^2 + \cdots, \tag{3.6}$$

showing that $\kappa(\theta; r, f) = O(1/r)$ for all $f \in \mathcal{S}$. It is clear that the smallest value of the coefficient of r in the expansion (3.6) is -3, and that it is attained when $|a_2| = 2$ and $\text{Re}\{(a_2^2 - a_3)e^{2i\theta}\} = 1$. Since this combination of coefficients is assumed only by the function

$$f(z) = k_{-\theta}(z) = \frac{z}{(1 - e^{-i\theta}z)^2} = z + 2e^{-i\theta}z^2 + 3e^{-2i\theta}z^3 + \cdots,$$

it is reasonable to suggest that $\kappa(\theta; r, k_{-\theta}) \leq \kappa(\theta; r; f)$ for small values of r and for all $f \in \mathcal{S}$, with equality holding if and only if $f(z) = k_{-\theta}(z)$. Since $\kappa(\theta; r, k_{-\theta}) = \kappa(0; r, k_0)$ by the rotation properties given in §2, we may rephrase this observation and formulate the following

Minimal Conjecture for the Curvature $\kappa(\theta; r, f)$. For small values of r,

$$\kappa(r) = \min_{f \in \mathcal{S}} \kappa(\theta; r, f) = \kappa(0; r, k_0) = \frac{1}{2r} \frac{\sqrt{\sum_{i=0}^{16} c_i r^i}}{(1 - r^2)^3} \tag{3.7}$$

where $c_0 = c_{16} = 1$,
$c_1 = c_{15} = 0$,
$c_2 = c_{14} = -18$,
$c_3 = c_{13} = 32$,

$c_4 = c_{12} = 97,$
$c_5 = c_{11} = -336,$
$c_6 = c_{10} = 330,$
$c_7 = c_9 = 452,$
and
$c_8 = -836.$

It is also clear that the largest value of the coefficient of r in the expansion (3.6) is $+2$, and that it is attained when $a_2 = 0$ and $\operatorname{Re}\{(a_2^2 - a_3)e^{2i\theta}\} = -1$. Since this combination of coefficients is assumed only by the function

$$f(z) = \sqrt{k_{-2\theta}(z^2)} = \frac{z}{(1-(e^{-i\theta}z)^2)} = z + e^{-2i\theta}z^3 + \cdots = h_{-\theta}(z)$$

it seems reasonable to suggest that $\kappa(\theta; r, f) \leq \kappa(\theta; r, \sqrt{k_{-2\theta}(z^2)}\,) = \kappa(\theta; r, h_{-\theta})$ for small values of r and for all $f \in \mathcal{S}$, with equality holding if and only if $f(z) = h_{-\theta}(z)$. Since $\kappa(\theta; r, h_{-\theta}) = \kappa(0; r, h_0) = \kappa(\frac{\pi}{2}; r, h_{-\frac{\pi}{2}})$ by the rotation properties given in §2, we may rephrase this observation and formulate the following

Maximal Conjecture for the Curvature $\kappa(\theta; r, f)$. For small values of r,

$$K(r) = \max_{f \in \mathcal{S}} \kappa(\theta; r, f) = \kappa(\frac{\pi}{2}; r, h_{-\frac{\pi}{2}}) = \frac{1}{2r} \frac{\sqrt{\sum_{i=0}^{16} \gamma_i r^i}}{(1+r^2)^3} \tag{3.8}$$

where $\gamma_0 = \gamma_{16} = 1,$
$\gamma_1 = \gamma_3 = \gamma_5 = \gamma_7 = \gamma_9 = \gamma_{11} = \gamma_{13} = \gamma_{15} = 0,$
$\gamma_2 = \gamma_{14} = 14,$
$\gamma_4 = \gamma_{12} = 49,$
$\gamma_6 = \gamma_{10} = 50,$
and
$\gamma_8 = 28.$

The Curvature $\kappa(r; \theta, f)$.

An elementary calculation shows that

$$\mathcal{Z}_\theta(r, \theta; f) = \sum_{n=0}^{\infty} e'_n(\theta) r^n,$$

and that

$$\mathcal{Z}_\theta^2(r, \theta; f) = \sum_{n=0}^{\infty} H_n(\theta) r^n,$$

where

$$H_n(\theta) = \sum_{i+j=n} e'_i(\theta) e'_j(\theta). \quad (n \geq 0)$$

Hence, from (1.9), we obtain

$$\kappa(r;\theta,f) = \sqrt{1+\frac{e_0'^2(\theta)}{4}} + \frac{1}{2}\frac{e_0'(\theta)e_1'(\theta)}{\sqrt{r+e_0'^2(\theta)}}r + \cdots$$

Since $e_0'(\theta) = 2\mathrm{Im}\{a_2e^{i\theta}\}$, and $e_1'(\theta) = 6\mathrm{Im}\{(a_3-a_2^2)e^{2i\theta}\}$, we get

$$\kappa(r;\theta,f) = \sqrt{1+(\mathrm{Im}\{a_2e^{i\theta}\})^2} + 3\left(\frac{\mathrm{Im}\{a_2e^{i\theta}\}\mathrm{Im}\{(a_3-a_2^2)e^{2i\theta}\}}{\sqrt{1+(\mathrm{Im}\{a_2e^{i\theta}\})^2}}\right)r + \cdots \tag{3.9}$$

It is clear that the largest value of the constant term in the expansion (3.9) is $+\sqrt{5}$, and that it is attained when $a_2 = \pm 2ie^{-i\theta}$. Since a_2 assumes these values only for the functions

$$\kappa_{\pm\frac{\pi}{2}-\theta}(z) = \frac{z}{(1\pm ie^{-i\theta}z)^2} = \frac{z}{\left(1-e^{i(\pm\frac{\pi}{2}-\theta)}z\right)^2},$$

it seems reasonable to suggest that $\kappa(r;\theta,f) \le \kappa(r;\theta,k_{\pm\frac{\pi}{2}-\theta})$ for small values of r and for all $f\in\mathcal{S}$, with equality holding if and only if $f(z) = k_{\pm\frac{\pi}{2}-\theta}(z)$. Since $\kappa(r;\theta,k_{\pm\frac{\pi}{2}-\theta}) = \kappa(r;\pm\frac{\pi}{2},k_0)$ by the rotation properties given in §2, we may rephrase this observation and formulate the following

Maximal Conjecture for the Curvature $\kappa(r;\theta,f)$. For small values of r,

$$K(\theta) = \max_{f\in S}\kappa(r;\theta,f) = \kappa(r;\pm\frac{\pi}{2},k_0) = \sqrt{\frac{\sum_{i=0}^{12} d_i r^i}{(1+r^2)^3}}$$

where $d_0 = d_{12} = 5$,
$d_1 = d_3 = d_5 = d_7 = d_9 = d_{11} = 0$,
$d_2 = d_{10} = 30$,
$d_4 = d_8 = 75$,
and
$d_6 = 96$.

There is no need to formulate a *Minimal Conjecture for the Curvature* $\kappa(r;\theta,f)$, since it has already been pointed out in §2 that this minimal problem has a trivial solution.

The Torsion $\tau(\theta;r,f)$.

An elementary calculation shows that

$$rZ_{r\theta}(r,\theta;f) = \sum_{n=1}^{\infty} ne_n'(\theta)r^n.$$

Placing this expansion and the previously derived expansions for $Z(r,\theta;f)$ and $\kappa^2(\theta;r,f)$ into (1.6), we obtain

$$\tau(\theta;r,f) = e_1'(\theta)r^2 + \big(e_0(\theta)e_1'(\theta) + 2e_2'(\theta)\big)r^3 + \cdots, \tag{3.10}$$

showing that $\tau(\theta; r, f) = O(r^2)$ for all $f \in \mathcal{S}$. It is clear that the largest value of the coefficient r^2 in the expansion (3.10) is $+6$, since

$$\begin{aligned} e_1'(\theta) &= 3\frac{d}{d\theta}\mathrm{Re}\{(a_2^2 - a_3)e^{2i\theta}\} \\ &= 3\mathrm{Re}\{2i(a_2^2 - a_3)e^{2i\theta}\} \\ &= 6\mathrm{Im}\{(a_3 - a_2^2)e^{2i\theta}\}, \end{aligned}$$

and it is possible that $(a_3 - a_2^2)e^{2i\theta} = i$. Earlier in this section of this paper, it was noted that if $|a_3 - a_2^2| = |b_1| = 1$, then $f(z)$ must assume the form (3.4). Since $a_3 - a_2^2 = -\overline{\eta}\,\overline{\xi}$, it must be the case that η and ξ satisfy the relation $\eta\xi = ie^{2i\theta}$, but this relationship alone does not determine the optimal choice for η and ξ. To make the correct choice for these constants, we must consider another term in the expansion (3.10) of $\tau(\theta; r, f)$; i.e., we must determine which choices of η and ξ cause the coefficient of r^3 to be maximal. A lengthy but straightforward calculation shows that

$$\begin{aligned} 2e_2'(\theta) + e_0(\theta)e_1'(\theta) = &-48\mathrm{Re}\{a_2e^{i\theta}\}\mathrm{Im}\{(a_3 - \tfrac{1}{2}a_2^2)e^{2i\theta}\} \\ &+12\mathrm{Im}\{(2a_4 - 3a_2a_3 + 2a_2^3)e^{3i\theta}\} + 12\mathrm{Im}\{a_3\overline{a_2}e^{i\theta}\}. \end{aligned}$$

If we let $\eta = ie^{i(\theta-\delta)}$ and $\xi = e^{i(\theta+\delta)}$, where δ is yet to be determined, then we obtain

$$2e_2'(\theta) + e_0(\theta)e_1'(\theta) = -24(\sin\delta + \cos\delta),$$

and this quantity is clearly maximized by the choice $\delta = -\frac{3\pi}{4}$. Consequently, we choose $\eta = ie^{i(\theta+\frac{3\pi}{4})}$, and $\xi = e^{i(\theta-\frac{3\pi}{4})}$. In turn, this implies that $f(z)$ must be of the form

$$f(z) = \frac{z}{\left(1 - e^{i(\frac{3\pi}{4}-\theta)}z\right)^2} = k_{\frac{3\pi}{4}-\theta}(z).$$

It now seems reasonable to suggest that $\tau(\theta; r, f) \le \tau(\theta; r, k_{\frac{3\pi}{4}-\theta})$ for small values of r and for all $f \in \mathcal{S}$, with equality holding if and only if $f(z) = k_{\frac{3\pi}{4}-\theta}(z)$. Since $\tau(\theta; r, k_{\frac{3\pi}{4}-\theta}) = \tau(\frac{3\pi}{4}; r, k_0)$ by the rotation properties given in §2, we may rephrase this observation and formulate the following

Maximal Conjecture for the Torsion $\tau(\theta; r, f)$. For small values of r,

$$\begin{aligned} T(r) &= \max_{f\in\mathcal{S}} \tau(\theta; r, f) \\ &= \tau(\frac{3\pi}{4}; r, k_0) \\ &= \frac{6r^2(1 - r^4)(1 + 2\sqrt{2}r + 5r^2 + 2\sqrt{2}r^3 + r^4)^2}{\left(\sum_{i=0}^{16} c_i r^i\right)} \end{aligned}$$

where $c_0 = c_{16} = 1$,
$c_1 = c_{15} = 0$,

$c_2 = c_{14} = -6,$
$c_3 = c_{13} = -8\sqrt{2},$
$c_4 = c_{12} = 25,$
$c_5 = c_{11} = 24\sqrt{2},$
$c_6 = c_{10} = 54,$
$c_7 = c_9 = -40\sqrt{2},$
and
$c_8 = -116.$

There is no need to formulate a *Minimal Conjecture for Torsion* $\tau(\theta; r, f)$ since it has already been pointed out in §2 that minimal value of the torsion is equal to the negative of maximal value.

The Torsion $\tau(r; \theta, f)$.

Substituting the previously derived expansions for $\mathcal{Z}(r, \theta; f)$, $r\mathcal{Z}_{r\theta}(r, \theta; f)$ and $\kappa^2(r; \theta, f)$ into (1.10), we obtain the expansion

$$\tau(r; \theta, f) = -\left(\frac{e_1'(\theta)}{4 + e_0'^2(\theta)}\right) + \left(\frac{2e_0'(\theta)e_1'(\theta)}{(4 + e_0'^2(\theta))^2} - \frac{e_0(\theta)e_1'(\theta) + 2e_2'(\theta)}{(4 + e_0'^2(\theta))}\right) r + \cdots$$

Since the constant term in this expansion may be expanded to be

$$-\left(\frac{e_1'(\theta)}{4 + e_0'^2(\theta)}\right) = -\frac{3}{2}\frac{\operatorname{Im}\{(a_3 - a_2^2)e^{2i\theta}\}}{1 + (\operatorname{Im}\{a_2 e^{i\theta}\})^2},$$

it is clear that its largest value is $+\frac{3}{2}$, and that it is attained by a function of the form (3.4), provided that $\operatorname{Im}\{a_2 e^{i\theta}\} = 0$, and that $\operatorname{Im}\{(a_3 - a_2^2)e^{2i\theta}\} = -1$. These requirements will be met if $\eta\xi = -ie^{2i\theta}$, and $(\overline{\xi} + \overline{\eta})e^{i\theta}$ is real. If we let $\eta = -ie^{i(\theta-\delta)}$ and $\xi = e^{i(\theta+\delta)}$, where δ is to be determined, then $\operatorname{Im}\{a_2 e^{i\theta}\} = \cos\delta - \sin\delta$, so that either $\delta = \frac{\pi}{4}$, or $\delta = -\frac{3\pi}{4}$. If $\delta = \frac{\pi}{4}$, then (3.4) becomes

$$f(z) = \frac{z}{\left(1 + (e^{i(\frac{\pi}{4}-\theta)}z)^2\right)} = \frac{z}{\left(1 - (e^{-i(\frac{\pi}{4}+\theta)}z)^2\right)} = h_{-\frac{\pi}{4}-\theta}(z).$$

If $\delta = -\frac{3\pi}{4}$, we are led to the same function. It now seems reasonable to suggest that $\tau(r; \theta, f) \le \tau(r; \theta, h_{-\frac{\pi}{4}-\theta})$ for small values of r and for all $f \in \mathcal{S}$, with equality holding if and only if $f(z) = h_{-\frac{\pi}{4}-\theta}(z)$. Since $\tau(r; \theta, h_{-\frac{\pi}{4}-\theta}) = \tau(r; -\frac{3\pi}{4}, h_{\frac{\pi}{2}})$ by the rotation properties given in §2, we may rephrase this observation and formulate the following

Maximal Conjecture for the Torsion $\tau(r; \theta, f)$. For small values of r,

$$T(\theta) = \max_{f \in \mathcal{S}} \tau(\theta; r, f) = \tau(r; \frac{\pi}{4}, h_0) = \frac{3}{2}\left(\frac{(1 - r^4)(1 + r^2 + r^4)^2}{(\sum_{i=0}^{12} \epsilon_i r^i)}\right),$$

where $\epsilon_0 = \epsilon_{12} = 1$,
$\epsilon_1 = \epsilon_3 = \epsilon_5 = \epsilon_6 = \epsilon_7 = \epsilon_9 = \epsilon_{11} = 0$,
and
$\epsilon_2 = \epsilon_4 = \epsilon_8 = \epsilon_{10} = 9$.

There is no need to formulate a *Minimal Conjecture for the Torsion* $\tau(r;\theta,f)$, since it has already been pointed out in §2 that the minimal value of the torsion is equal to the negative of the maximal value.

4. Open Problems

1. All of the extremal curvature and torsion problems posed in §2 remain unsolved, except for the partial solutions presented for the two minimal curvature problems.

Variational methods provide some additional information. Let $f \in \mathcal{S}$, and let $f^* \in \mathcal{S}$ be a variation of f; i.e., let $f^*(z) = f(z) + \epsilon g(z) + o(\epsilon)$. Then

$$\mathcal{Z}(r,\theta;f^*) = \mathcal{Z}(r,\theta;f) - 2\epsilon \mathrm{Re}\Psi(g(z);f(z)) + o(\epsilon), \tag{4.1}$$

where

$$\Psi(g(z);f(z)) = \frac{\left(\frac{g'(z)}{f'(z)}\right) - \left(\frac{2|f(z)|^2}{1+|f(z)|^2}\right)\left(\frac{g(z)}{f(z)}\right)}{2rf^{\#}(z)} \tag{4.2}$$

Note that $\Psi(zf'(z);f(z))$ is actually the same as the quantity $\Phi(z;f)$, which was defined by (1.12).

Variational formulas for all of the curvature and torsion formulas may be expressed in terms of $\Psi(g(z);f(z))$. For example,

$$\kappa(\theta;r,f^*) = \kappa(\theta;r,f) + \frac{\epsilon}{\kappa(\theta;r,f)}[\mathrm{Re}\Psi(zf'(z);f(z)]\left[r\frac{\partial}{\partial r}\mathrm{Re}\Psi\left(g(z);f(z)\right)\right] + o(\epsilon).$$

If all four curvature problems are considered as one, then, after a careful derivation, the corresponding quadratic differential becomes

$$Q(w)dw^2 = \eta\left(\frac{Aw^2+Bw+C}{(w-r)^3}\right)\frac{dw^2}{w^2},$$

where $|\eta| = 1$, and A, B, and C are constants which depend upon the solution $f(z)$ to the problem.

2. New subclasses of $\mathcal{S}$ may be defined. For example, for a given r and θ, we may define

$$\mathcal{S}_T(r,\theta) = \{f \in \mathcal{S} : |\tau(\theta;r,f)| \le T\}$$

to be the class of functions with *Bounded Spherical Torsion T*. Then, extremal problems, e.g., coefficient estimation problems, may be posed for them.

3. Since univalence criteria, given in terms of the Schwarzian derivative $\{f, z\}$ exist, [2, pp. 261,264], it may be entirely possible to provide additional univalence criteria in terms of restrictions upon the curvature and torsion functions.

4. If $f \in \mathcal{S}$ has a pole at a point $\zeta \in \partial\mathcal{U}$, how do the curvature and torsion functions behave as $z = re^{i\theta} \to \zeta$?

5. Do there exist maximum principles for the curvature and torsion functions? For example, if $0 < r_1 < r_2 < 1$, does the maximum value of the curvature $\kappa(\theta; r, f)$ occur on the boundary of the annulus $A(r_1, r_2) = \{z : r_1 < |z| < r_2\}$?

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