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# COMMUTATIVITY AND DECOMPOSITION FOR NEAR RINGS

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**Abstract.** Let $R$ be a distributively generated (d.g) near ring satisfy one of the following conditions.

(*) For each $x, y$ in $R$, there exists a positive integer $n = n(x, y)$ such that $xy = (yx)^n$.

(**) For each $x, y$ in $R$, there exist positive integers $m = m(x, y)$ and $n = n(x, y)$ for which $xy = y^m x^n$.

In [2], Bell proved the commutativity of $R$ satisfying (*) or (**) under appropriate additional hypothesis. In this paper, we generalize the above properties for wider class of near rings known as D-near rings. Also we provide an example for justification of our results. Furthermore, we give a decomposition Theorem for near rings satisfying (**).

## 1. Some Commutativity Theorems for Near Rings

In [6], Ligh and Luh introduced the notion of D-ring as follows:

**Definition 1.** A near ring $R$ is called a D-near ring if every non-zero homomorphic image $S$ of $R$ satisfies the following conditions:

$(C_1)$ $S$ has a non-zero right distributive element.

$(C_2)$ $(S, +)$ is abelian implies that $(S, +, .)$ is a ring.

All rings and distributively generated ($d.g$) near rings are examples of D-near rings. The following example shows that D-near rings are generalizations of $d.g$ near rings.

**Example 1.** Let $R = \{0, x, y, z, u, v\}$ with addition and multiplication tables, defined as follows:

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| + | 0 | x | y | z | u | v |
|---|---|---|---|---|---|---|
| 0 | 0 | x | y | z | u | v |
| x | x | 0 | v | u | z | y |
| y | y | u | 0 | v | x | z |
| z | z | v | u | o | y | x |
| u | u | y | z | x | v | 0 |
| v | v | z | x | y | 0 | u |

| · | 0 | x | y | z | u | v |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| x | 0 | x | x | x | 0 | 0 |
| y | 0 | x | z | y | v | u |
| t | 0 | x | y | z | u | v |
| u | 0 | 0 | 0 | 0 | 0 | 0 |
| v | 0 | 0 | 0 | 0 | 0 | 0 |

Then $R$ is a D-near ring which has a unique left identity $z$ with $uz = 0 = vz$. This indicates that $z$ is not a right identity. By virtue of [7, Theorem 3.2], if a $d.g$ near ring has a unique left identity, then it is also a right identity. Thus $R$ is not a $d.g$ near ring. This shows that the class of D-near rings is larger than the class of $d.g$ near rings.

Let $R$ be a D-near ring. In this section, we study the commutativity of $R$ satisfying one of the following conditions:

(*) For each $x, y$ in $R$, there exists a positive integer $n = n(x, y)$ such that $xy = (yx)^n$.
(**) For each $x, y$ in $R$, there exist positive integers $m = m(x, y)$ and $n = n(x, y)$ for which $xy = y^m x^n$.

**Definition 2.** A near ring $R$ is called zero-symmetric if $0x = 0$ for all $x \in R$, that is left distributive gives $x0 = 0$.

**Definition 3.** A near ring $R$ is called zero-commutative if $xy = 0$ implies that $yx = 0$ for $x, y \in R$.

First, we recall [1, Lemma 1], [1, Lemma 2] and [3, Lemma 3].

**Lemma 1.** *Let $R$ be a zero-symmetric near ring satisfying the following conditions:*

(a) *For each $x$ in $R$, there exists an integer $n = n(x) > 1$ such that $x^n = x$.*
(b) *Every non-trivial homomorphic image of $R$ contains a non-zero central idempotent. Then $(R, +)$ is abelian.*

**Lemma 2.** *Let $R$ be a zero-symmetric near ring with no non-zero nilpotent elements. Then the following conditions hold.*

(i) *Every distributive idempotent is central.*
(ii) *For every idempotent $e$ and every element $y$ in $R$, $ey^2 = (ey)^2$.*
(iii) *If $R$ has a multiplicative identity element, then all idempotent elements are central.*

**Lemma 3.** *Let $R$ be a near ring which is zero commutative. Then*

(i) *If $a, b \in R$ such that $ab = 0$, then $arb = 0$ for all $r \in R$.*
(ii) *The annihilator of any non-empty subset of $R$ is an ideal.*

(iii) *The set of all nilpotent elements is an ideal if it is a subgroup of the additive group $R^+$ of $R$.*

First, we prove the following lemma.

**Lemma 4.** *Let $R$ be a near ring satisfying $(*)$ or $(**)$. Then the idempotent elements of $R$ are central.*

**Proof.** Let $R$ satisfies $(*)$, and let $e$ be an idempotent element of $R$. If $x \in R$, then there exist integers $p = p(e,x) \geq 1$ and $n = n(x,e) \geq 1$ such that $xe = (ex)^n$ and $ex = (xe)^p$. Multiplying by $e$ on the left of the first and right of the second, we get $exe = e(ex)^n = (ex)^n = xe$ and $exe = (xe)^p e = (xe)^p = ex$. Thus, $ex = xe$. Therefore, the idempotent elements of $R$ are central.

Now, let $R$ satisfies $(**)$ and $e$ be an idempotent element of $R$. Then there exist integers $r = r(x,e) \geq 1$ and $s = s(x,e) \geq 1$ such that $xe = e^r x^s = ex^s$. Thus $exe = ex^s = xe$. Also, for some $m = m(e,x) \geq 1$ and $n = n(e,x) \geq 1$, we have $ex = x^m e^n = x^m e$. Thus $exe = x^m e = ex$. So $exe = ex$. Hence $ex = xe$ for all $x$ in $R$. Therefore, the idempotent elements of $R$ are central.

**Lemma 5.** *Let $R$ be a zero-symmetric D-near ring. If for each $x$ in $R$, there exists a positive integer $m = m(x) > 1$ such that $x^m = x$, then $R$ is a commutative ring.*

**Proof.** By the definition of D-near ring, every non-zero homomorphic image of $R$ contains a non-zero distributive element. If $x$ is a non-zero distributive element with $m = m(x) > 1$ such that $x^m = x$, then $x^{m-1} = e$, that is $x^{m-1}$ is a distributive idempotent. In view of Lemma 2(i), $x^{m-1}$ is a distributive central idempotent, because $R$ has no non-zero nilpotent elements. By Lemma 1, $(R, +)$ is abelian. But $R$ is a D-near ring. Hence $R$ is a ring. By a well-known result of Jacobson [5], $R$ is a commutative ring.

Now, we are in a position to prove our main results.

**Theorem 1.** *Let $R$ be a D-near ring satisfying $(*)$. Thus $R$ is commutative.*

**Proof.** If $n = n(x,y) = 1$, then $xy = yx$. Thus $R$ is commutative. Assume that $n = n(x,y) > 1$ and $R$ satisfies the property $(*)$. First, we show that $xy = 0$. By $(*)$, we have $yx = (xy)^n = 0$. Therefore, $R$ is zero-commutative. Thus the left and the right annihilator of $R$ coincide. Let $A$ be an an annihilator of $R$. By Lemma 3(ii), $A$ becomes an ideal. Let $a$ in $R$ with $a^2 = 0$. Given any $x \in R$ we have $axa = 0$ because $a(ax) = 0$ and $R$ is zero-commutative. By assumption $ax = (xa)^n$ for some $n = n(a,x) > 1$. Therefore $ax = x(axa)(xa)^{n-2} = 0$ and so $a \in A$. Since the homomorphic image $R/A$ of $R$ is a D-near ring, take positive integers $p$ and $q$ such that $x^2 = x^{p+q}$. Thus $x(x^{p+q-1} - x) = 0$. But $R$ is zero-commutative. So $(x^{p+q-1} - x)x = 0$. Hence $x(x^{p+q-1} - x)x^{p+q-1} = 0$. Therefore, $(x^{p+q-1} - x)^2 = 0$. This shows that $x^{p+q-1} - x \in A$. By Lemma 5, $R/A$ is a commutative ring. Hence $x(xy - yx) = 0$, and so $x^2 y = xyx$ for all $x \in R$. But

$x^2 = x^{p+q}$. Thus $x^{p+q-2} = e$ is idempotent and so is central by Lemma 4. Hence

$$\begin{aligned} yx^2 &= x^{p+q-2}yx^2 \\ &= x^{p+q-3}xyx^2 \\ &= x^{p+q-3}(xyx)x \\ &= x^{p+q-3}x^2yx \\ &= x^{p+q-2}xyx \\ &= x^{p+q-2}x^2y \\ &= x^2y. \end{aligned}$$

So

$$yx^2 = x^2y = xyx \text{ for all } x, y \in R. \tag{1}$$

By our hypothesis (*), we can write

$$xy = (yx)^n \text{ where } n = n(x, y) > 1.$$

Similarly, for each pair of elements $y, x$ in $R$, there exists an integer $r = r(y, x) > 1$ such that $yx = (xy)^r$. This implies that $(xy)^{rn} = xy$. Now, we have

$$\begin{aligned} xy &= (xy)^{rn} \\ &= \big((xy)^r\big)^n \\ &= (\underbrace{xy \cdot xy \cdot xy \cdots xy}_{r-times})^n \\ &= \big((xyx)\underbrace{yx \cdot yx \cdot yx \cdots yx}_{(r-2)-times}\, y\big)^n \end{aligned}$$

Repeated (1) continuously, we get

$$\begin{aligned} xy &= (yx^2 \underbrace{yxyx \cdots yx}_{(r-2)-times}\, y)^n \\ &= \big((yx)^2 \underbrace{xyxy \cdots xy}_{(r-2)-times}\big)^n \\ &= \big((yx)^{r-2}yx^2y\big)^n \\ &= \big((yx)^{r-2}yxyx\big)^n \\ &= \big((yx)^r\big)^n \\ &= (xy)^n \\ &= yx. \end{aligned}$$

Therefore, $R$ is commutative.

**Theorem 2.** *Let $R$ be a D-near ring satisfying $(**)$. Then $R$ is commutative.*

**Proof.** Let $R$ satisfies $(**)$ and let $n = n(x,y) \geq 1$ and $m = m(x,y) \geq 1$. Then it is easy to check that $R$ is zero-commutative. Then by $(**)$, we get

$$yx = x^n y^m = \underbrace{xx \cdots (x}_{n-times} \underbrace{y)y \cdots y}_{m-times} = 0.$$

Let $x, y \in R$ with $xy = 0$. Then $R$ is zero commutative.

Using the same argument as in the proof of Theorem 1, we get

$$x^2 y = xyx = yx^2 \text{ for all } x, y \in R.$$

This implies that for any $s \geq 2$, we can write

$$x^s y = yx^s \text{ for all } x, y \in R. \tag{2}$$

Now by $(**)$, we have $xy = y^m x^n$. Let $y = x^p y^q$ for some positive integers $p$, $q$, $m$ and $n$. Using (2), we obtain

$$\begin{aligned} xy &= y^{m-1} yx x^{n-1} \\ &= y^{m-1} x^p y^q x^{n-1} \\ &= y^{m+q-1} x^{n+p-1}. \end{aligned}$$

Thus

$$xy = y^{m+q-1} x^{n+p-1}. \tag{3}$$

Furthermore, by using (2) and (3), we get

$$\begin{aligned} xy &= x^{n+p-1} y^{m+q-1} \\ &= x^{p-1} x^n y^m y^{q-1} \\ &= x^{p-1} y^m x^n y^{q-1} \\ &= x^p y^q \quad = yx. \end{aligned}$$

Hence $xy = yx$. Therefore, $R$ is commutative.

## 2. A Decomposition Theorem for Near Rings

In [4], Bell and Ligh established the direct sum decomposition for rings satisfying the properties $xy = (xy)^2 f(xy)$ and $xy = (yx)^2 f(yx)$, where $f(X) \in \mathbb{Z}[X]$, the polynomial ring over $\mathbb{Z}$. Furtheremore, in [4], they remarked that in case of near rings the analogous results do not give direct sum decomposition. The authors of [4], defined a weaker condition of orthogonal sum as follows.

**Definition 4.** A near ring $R$ is an orthogonal sum of subnear rings $M$ and $N$ denoted by $R = M \oplus N$, if $MN = NM = (0)$ and each element of $R$ has a unique representation in the form $m + n$ such that $m \in M$ and $n \in N$.

In this paper, we consider the near ring property:

$(***)$ For each $x, y$ in a near ring $R$, there exist positive integers $m = m(x,y) \geq 1$ and $n = n(x,y) > 1$ for which $xy = y^m x^n$. Indeed, we prove the following result.

**Theorem 3.** *Let $R$ be a near ring satisfying $(***)$. Then the set $N$ of all nilpotent elements of $R$ is a subnear ring with trivial multiplication. Indeed $N$ is an ideal in $R$. If $M = \{x \in R | x^n = x$, for a positive integer $n(x) > 1\}$, then $M$ is a subnear ring of $R$ with $(M, +)$ is abelian. Furthermore, $R = M \oplus N$.*

Before proving our decomposition result for near rings, we state the following lemma.

**Lemma 6** [4]. *Let $R$ be a near ring with idempotent elements are multiplicative central, and let $e$ and $f$ be any idempotent element of $R$. Then there exists an idempotent element $g$ such that $ge = e$ and $gf = f$.*

**Proof of Theorem 3.** Let $R$ satisfies $(***)$. Then it is easy to check that $R$ is necessarily zero-symmetric as well as zero-commutative. Suppose that $a \in N$ and $x \in R$. Then there exist integers $m_1 = m_1(a,x) \geq 1$, and $n_1 = n_1(a,x) > 1$ such that

$$ax = x^{m_1} a^{n_1}. \tag{4}$$

Now, select $m_2 = m_2(x^{m_1}, a^{n_1}) \geq 1$ and $n_2 = n_2(x^{m_1}, a^{n_1}) > 1$ such that

$$x^{m_1} a^{n_1} = a^{n_1 n_2} x^{m_1 m_2}. \tag{5}$$

Combining (4) and (5), we obtain

$$ax = a^{n_1 n_2} x^{m_1 m_2}$$

Using the same argument as above for arbitrary $q$, such that the integers $m_1 \geq 1$, $m_2 \geq 1$, ..., $m_q \geq 1$ and $n_1 > 1$, $n_2 > 1$, ..., $n_q > 1$. But $a \in N$. So $a^{n_1 n_2 \cdots n_q} = 0$ for sufficiently large $q$. Thus $ax = 0$. But $R$ is zero-commutative. Then the nilpotent elements of $R$ annihilate $R$ on both sides, so $NR = RN = (0)$. This implies that $N^2 = (0)$ and $N \subseteq Z(R)$, the center of $R$. Further, let $a, b \in N$ such that $a^{s'} = 0$ and $b^{t'} = 0$ for all $s' \geq 1$ and $t' > 1$. Then $(a-b)^{s'+t'} = 0$, that is $a - b \in N$. By Lemma 3 (iii), we get $N$ is an ideal of $R$.

Let $r \in R$ and let $s > 1$, $t > 1$ be integers such that $r^{s+t} = r^2$. So we have $r = r - r^{s+t-1} - r^{s+t-1}$. Because $r(r - r^{s+t-1}) = 0$ and $R$ is zero-commutative. So we get $(r - r^{s+t-1})r = 0$ and $(r - r^{s+t-1})r^{s+t-1} = 0$. Hence $(r - r^{s+t-1})^2 = 0$ and $r - r^{s+t-1} \in N$. Also, we have

$$\begin{aligned}
(r^{s+t-1})^{s+t-1} &= r^{(s+t-1)(s+t-1)} \\
&= r^{(s+t-2)(s+t)} \cdot r \\
&= (r^{s+t})^{s+t-2} \cdot r \\
&= (r^2)^{s+t-2} \cdot r \\
&= (r^{s+t-2})^2 \cdot r.
\end{aligned}$$

Since $r^{s+t-2}$ is idempotent, $(r^{s+t-1})^{s+t-1} = r^{s+t-1}$ for $s+t-1 > 1$ and $r^{s+t-1} \in M$.

Next, we show that $M$ is a subnear ring of $R$. Let $u, v \in M$ and let $l = l(u) \geq 1$, $k = k(v) > 1$ be integers such that $u^l = u$ and $v^k = v$. Then $e = u^{l-1}$ and $f = v^{k-1}$ are idemoptent elements such that $eu = u$ and $fv = v$. Thus

$$uv = eufv = efuv = uvef = (ef)^m (uv)^n$$

for some integers $m = m(uv, ef) \geq 1$ and $n = n(uv, ef) > 1$. Hence, we can write

$$uv = ef(uv)^n = (uv)^n.$$

This implies that $uv \in M$. Since $R/N$ has the property $x^{j(x)} = x$ for an integer $j(x) > 1$. So we have an integer $i > 1$ such that

$$(u-v)^i = (u - v + a) \text{ for } a \in N. \tag{6}$$

Using Lemma 6, we choose an idempotent $h$ for which $he = e$ and $hf = f$ such that $hu = u$, and $hv = v$. Multiplying (6) by $h$, we get $(u-v)^i = u - v \in M$. This implies that $M$ is a subnear ring.

By Lemma 1, $(M, +)$ is abelian. It is obvious to see that $M \cap N = (0)$. Now, let $x_1 + y_1 = x_2 + y_2$ for $x_1, x_2 \in M$ and $y_1, y_2 \in N$. Then $x_1 - x_2 = y_2 - y_1$. Since $x_1 - x_2 \in M$ and $y_2 - y_1 \in N$. This implies that $x_1 - x_2 = y_2 - y_1 \in M \cap N = (0)$. Thus $x_1 = x_2$ and $y_1 = y_2$. Therefore, $R = M \oplus N$.

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