TAMKANG JOURNAL OF MATHEMATICS
Volume 28, Number 2, Summer 1997

# ON P-HARMONIC MAPS AND THEIR APPLICATIONS TO GEOMETRY, TOPOLOGY AND ANALYSIS

SHIHSHU WALTER WEI

## 0. Introduction

The use of the $p$-energy functional on spaces of maps $u : M \to N$ between Riemannian manifolds has contributed to our understanding of mathematics. The critical points of the $p$-energy functional, $p$-harmonic maps, have been employed in many different contexts for solving various problems.

(i) For $p = 1$, Bombieri-De Giorgi-Giusti [3] construct a 1-harmonic function $u : R^{2m} \to R$ whose zero-level set $u^{-1}(0) = \{x \in R^{2m} : x_1^2 + \cdots + x_m^2 = x_{m+1}^2 + \cdots + x_{2m}^2\}$ is an area-minimizing cone over the product of $(m-1)$-spheres $\{x \in R^{2m} : x_1^2 + \cdots + x_m^2 = x_{m+1}^2 + \cdots + x_{2m}^2 = \frac{1}{2}\}$ in $R^{2m}$ for $m \geq 4$.

Arising also from a nonlinear entire solution [3] of the minimal surface equation by means of the Fleming-De Giorgi construction ([20], [10]), this cone provides the first counter-example to interior regularity for solutions to the co-dimension 1 Plateau problem, and shows the optimality of the estimate $n-7$ obtained by Federer's Reduction technique for the highest possible Hausdorff dimension of the singular set of an $n$-dimensional area-minimizing rectifiable current in $R^{n+1}$ [21].

In the paper of S.P. Wang and S.W. Wei [73], a class of counter-examples to a Bernstein-type theorem in hyperbolic space was constructed, using smooth 1-harmonic function in hyperbolic space. Since then many works on this topic were developed by M. Anderson (in his Berkeley thesis) [1], R.M. Hardt and F.H. Lin [31], and recently by others.

(ii) In the case $p = 2$, $p$-harmonic maps are harmonic maps which include harmonic functions (where $N = R$) and harmonic 1-forms (where $N = S^1$). The study of harmonic functions, which arises from complex analysis, stimulates the the development of the theory of elliptic and parabolic P.D.Es and serves as a tool in solving geometric and classical variational problems [9] [45], while the study of harmonic forms which uniquely represent cohomology classes according to Hodge Theory, has extensive applications to topology, analysis and geometry.

A pioneering theorem of Eells-Sampson [19] shows the existence of a harmonic map in the homotopy class of any smooth map on a compact manifold into a compact manifold

Received December 24, 1996.

with nonpositive sectional curvature. Hartman [28] proves the uniqueness of harmonic maps into compact manifolds with negative sectional curvature and rank $> 1$ somewhere. The Dirichlet problem for harmonic maps with compact, nonpositive target manifolds is solved by Hamilton [26], and the result is generalized to harmonic maps with images in geodesically small discs by Hildebrandt-Kaul-Widman [29].

(iii) For $1 < p < \infty$, $p$-harmonic maps include geodesics (where $\dim M = 1$) and minimal submanifolds $M$ in $N$ ( where $\dim M = p$). In fact, suppose $M$ is one-dimensional. Then for any $p \geq 1$, $u : M \to N$ is $p$-harmonic and parametrized proportionally to the arc length if and only if $u$ is a geodesic in $N$, and $u$ is $p$-energy minimizing if and only if $u$ is length-minimizing [68]. On the other hand, suppose $M$ is $m$-dimensional and $u : M \to N$ is an isometric immersion. Then for any $m > 1$, $u : M \to N$ is $m$-harmonic if and only if $M$ is a minimal submanifold in $N$, and $u$ is $m$-energy minimizing if and only if $M$ is area-minimizing [68]. In general, an isometric immersion $u : M \to N$ is $p$-harmonic for any $p \geq 1$ if and only if $u$ is minimal.

The idea of representing fundamental groups by closed geodesics is due to E. Cartan [8] who proves that if $M$ is a compact Riemannian manifold, then every nontrivial free homotopy class of loops in $M$ contains a closed geodesic of minimum length. The study of geodesics leads to many beautiful discoveries in Riemannian Geometry (such as the Cartan-Hadamard, Bonnet-Meyer, Synge, Rauch Comparison Theorems), Classical Morse Theory (where it plays an important role in solving the higher dimensional Poincaré conjecture [52]) and Morse Theory on infinite dimensional spaces such as Banach manifolds [44] [46] [50].

According to a theorem of Federer-Fleming, minimal varieties of least area represent homology groups. The theorem asserts that the singular homology groups of every compact Lipschitz neighborhood retract $A$ in $n$-space with integer coefficients are isomorphic with the homology groups of the complex of all integral currents with support in $A$; in each integral homology class there is a cycle of least mass [22]. Analogous to harmonic maps are minimal submanifolds, which have proven to be useful in applications to differential geometry, algebraic geometry, geometric measure theory, topology, partial differential equations, complex analysis, mathematical physics and representation theory.

(iv) The case $p = \infty$ has been studied by L.C. Evans [12].

## 1. Regularity Theory

Regularity estimates for elliptic systems, in particular the Euler-Lagrange equation for $p$-energy, were first obtained by K. Uhlenbeck [63] for $p \geq 2$, and later by P. Tolksdorf [61] for $p > 1$. The question of minimizing the $p$-energy in appropriate homotopy classes has been studied by B. White [71,72]. A ground-breaking regularity theory of $p$-energy minimizing maps between Riemannian manifolds has been established by Hardt-Lin [30] and Luckhaus [34], for case $p > 1$ (the case $p = 2$ is due to Schoen-Uhlenbeck [56] and Giaquinta-Giusti [24]).

Let $M^n$ be a compact Riemannian manifold with possibly non-empty boundary, and $N^k$ be isometrically immersed in $R^q$. $L_1^p(M,N)$ denotes the set of maps $u: M \to R^q$ whose component functions have first weak derivatives in $L^p$ and $u(x) \in N$ a.e. on $M$. The $p$-energy for $u \in L_1^p(M,N)$ is given by

$$E_p(u) = \frac{1}{p}\int_M |du|^p dx \tag{1.1}$$

where $du$ denotes the differential of $u$, $dx$ is the volume element of $M$ and $1 \le p < \infty$. A map $u \in L_1^p(M,N)$ is said to be **p-harmonic** if it is a weak solution to the Euler-Lagrange equation for $E_p$ on $L_1^p(M,N)$. $u$ is called **p-stable** (or **p-minimizing**) if $u$ is a local (resp. global) minimum of the $p$-energy functional $E_p$ on $L_1^p(M,N)$ having the same trace on $\partial M$. $u$ is said to be **p-unstable** if $u$ is not $p$-stable.

**1.2. Definition.** A map $\overline{u}: R^{j+1} \to N$ is said to be a **p-minimizing tangent map** ($p$-MTM) if $\overline{u}$ is $p$-minimizing on every compact subset of $R^{j+1}$ and is a homogeneous extension of $u: S^j \to N$ of degree-zero.

**1.3. Theorem (Hardt-Lin [30], Theorem 4.5 p.573).**
*Suppose $\ell$ is the largest integer such that any p-minimizing tangent map from the unit ball in $R^j$ into $N$ is a constant map for each $j = 1, \dots, \ell$. Then the interior singular set of any p-minimizer $u \in L^{1,p}(\Omega, N)$ is empty in case $n < \ell + 1$, is a discrete set in case $n = \ell + 1$, and has Hausdorff dimension $n - \ell - 1$ in case $n \geqq \ell + 1$. Moreover, $\ell \geqq [p]$. (Where $\Omega$ is a $C^2$ bounded open subset of $R^n$ with the Euclidean metric.)*

As a consequence,

**1.4. Theorem (Hardt-Lin [30], Luckhaus [34]).** *For $p > 1$, the Hausdorff dimension of the singular set of a p-minimizing map $u \in L_1^p(M^n, N^k)$ in the interior of $M$ cannot exceed $n - [p] - 1$, in general. If $n = [p] + 1$, $u$ has at most isolated singularities. If $n < [p] + 1$ (or if $[p] + 1 \le n$, off the singular set) $u$ is locally Hölder continuous up to the boundary and the gradient of $u$ is also locally Hölder continuous in the interior of $M$.*

Since then the regularity problem of $p$-minimizing maps between Riemannian manifolds has become an active research area. Furthermore, the above Theorems 1.3 and 1.4 also have an impact on the development of the theory of $p$-harmonic maps which we will discuss in subsequent sections (e.g. §2, 5, 6, 7, 9).

Generalizing the work of Ming Li [35] which treats the case of $p = 2$, Shah-Sen Wang proves some general partial interior regularity for $p$-minimizers into complete manifold $N$ ( which are not necessary the retract of a uniform tubular neighborhood $N_\tau$ ).

**1.5. Theorem [70].** *A p-minimizer $u \in L_1^p(M,N)$ is $C^{1,\alpha}$ outside a closed singular set $S_p$ in the interior of $M$ with Hausdorff dimension at most $n - p$.*

The regularity of $p$-energy minimizing section of fiber bundle has been studied by Shah-Sen Wang [69], in which the Hausdorff dimension of the singular set does not exceed $n-[p]-1$.

On the other hand, using Hardy Method (compensated compactness) in harmonic analysis, Libin Mou and Paul Yang have established the partial regularity of stationary $p$-harmonic maps into a sphere, thus generalizing results of L. C. Evans [11] for $p=2$.

**1.6. Theorem** [41]. *Let $M$ be a compact $n$-manifold. For $1<p\le n$, a stationary $p$-harmonic map $u \in L_1^p(M,S^k)$ is $C^{1,\alpha}$ outside a closed singular set $S_p$ with $n-p$ dimensional Hausdorff measure zero. If $p=n$, then $u$ is $C^{1,\alpha}$.*

More recently, T. Toro and C.Y. Wang [62] have shown that a stationary $p$- harmonic maps from an open subset of $R^n$ into a homogeneous space with a left invariant metri is $C^{1,\alpha}$ except possibly for a closed singular set of $n-p$ dimensional Hausdorff measure zero .

## 2. Fundamentals in Differential Geometry

In differential geometry, $p$-harmonic maps are natural objects of study [68]. We begin with length-minimizing geodesics:

**2.1. Proposition.** *Let $M$ be a complete Riemannian manifold, $x_1, x_2 \in M$ have distance $d$, and let $\Omega(M;x_1,x_2)$ be the set of all piecewise smooth paths from $x_1$ to $x_2$ in $M$. Then the $p$-energy function*

$$E_p : \Omega(M;x_1,x_2) \to R$$

*takes on its minimum $\frac{1}{p}d^p$ precisely on the set of length-minimizing geodesics from $x_1$ to $x_2$.*

**Proof.** Denote $L(u)$ the length of a curve $u(t)$ from $t=a$ to $t=b$. Then by Hölder inequality

$$\begin{aligned} L(u)^p &\le \Big(\int_a^b |\frac{du}{dt}|^p dt\Big)\Big(\int_a^b 1^q dt\Big)^{\frac{p}{q}} \quad \text{where} \quad \frac{1}{p}+\frac{1}{q}=1 \\ &\le (b-a)^{\frac{p}{q}} pE_p(u) \quad \text{and equality holds} \quad \text{if and only if} \quad |\frac{du}{dt}| \equiv c \end{aligned}$$

Hence, if $w$ is length-minimizing from $x_1 = w(0)$ to $x_2 = w(1)$, then

$$pE_p(w) = L(w)^p \le L(u)^p \le pE_p(u) \tag{2.2}$$

Here the equality $L(w)^p = L(u)^p$ can hold only if $u$ is also length-minimizing, possibly reparametrized. On the other hand, the equality $L(u)^p = pE_p(u)$ can hold only if the parameter is proportional to arc-length along $u$. This proves that $E_p(w) < E_p(u)$ unless $u$ is also length-minimizing.

Analogous to (2.2),

**2.3. Proposition.** *Let $u$ be an immersion of an $n$-manifold $M$ in $N$. Denote the $n$-dimensional area of $u(M)$ by $A(u)$. Then*

$$A(u) \leq n^{-\frac{n-2}{2}} E_n(u) \quad \textit{and equality holds if and only if} \quad u \quad \textit{is conformal}.$$

For the mass functional, the first and second variational formulas are due to Federer in flat space [20] and to Lawson-Simons in the general Riemannian case [36] ; formulas for the energy functional and for the area functional are found in Eells-Sampson [19] and in Simons [51] respectively. In [68] we develop fundamental tools in the study of $p$-harmonic maps by deriving the first and second variational formulas for the $p$-energy functional on maps between Riemannian manifolds, and obtaining estimates on a Bochner formula in this general setting.

Consider a $C^1$ map $u : M \to N$, where $M$ is compact. Denote the pull-back tangent bundle of $N$ by $u$ as $u^{-1}TN$, and the pull-back connection by $\nabla^u$.

Choose a one-parameter $C^2$ family of $C^1$ maps $u_t$ such that $u_0 = u$ and $\frac{du_t}{dt}|_{t=0} = v$ is $C^1$ and a two-parameter $C^1$ variations $F(\cdot, s, t) = u_{s,t}$ such that

$$V = \frac{\partial u_{s,t}}{\partial s}, \quad v = \frac{\partial u_{s,t}}{\partial s}|_{0,0}, \quad W = \frac{\partial u_{s,t}}{\partial t} \quad \text{and} \quad w = \frac{\partial u_{s,t}}{\partial t}|_{0,0}.$$

We shall also denote by $F : M \times [0,1] \to N$ the map defined by $F(x,t) = u_t(x)$ for one-parameter $C^2$ variations through $C^1$ maps, i.e. $F$ is $C^1$ in $x$ and $C^2$ in $t$.

**2.4. First variational formula for p-energy.**

$$\frac{d}{dt} E_p(u_t) = \int_M |du_t|^{p-2} \sum_i < \nabla^F_{e_i} V, F_*(e_i) > dx$$

*where $\{e_1, \cdots, e_n\}$ is a local orthonormal frame field on $M$, and $V = F_*(\frac{\partial}{\partial t})$.*

**2.5. Corollary.** *A curve $u$ is $p$-energy minimizing if and only if $u$ is length-minimizing. A curve $u$ is a $p$-harmonic and parametrized proportionally to the arc length if and only if $u$ is a geodesic.*

Proof. This follows directly from the first variational formula 2.4. and the proof of Proposition 2.1.

**2.6. Corollary.** *An isometric immersion $u : M \to N$ is $p$-harmonic for any $p \geq 1$ if and only if $u$ is minimal.*

Proof. Assume the dimension of $M$ is $n$. Then by setting $t = 0$ (2.4) in which identifying $e_i$ with $du(e_i)$, $|du|^{p-2} = n^{\frac{n-2}{2}}$ , it follows that

$$0 = \int_M \sum_i < \nabla^u_{e_i} |du|^{p-2} du(e_i), V > dx = \int_M n^{\frac{n-2}{2}} \sum_i < \nabla^N_{du(e_i)} du(e_i), V^\perp + V^T > dx$$

$$= \int_M n^{\frac{n-2}{2}} (< H, V > - divV^T >)dx = n^{\frac{n-2}{2}} \int_M < H, V > dx$$

where $H$ is the mean curvature of $M$ in $N$, $V^T$ and $V^\perp$ are tangential and normal components of $M$ in $N$ respectively. This completes the proof.

Dr. Ren-Long Xia has proved the above result 2.6 by the method of moving frame.

**2.7. Second variational formula of two parameters for p-energy.** ($u$ is not necessary $p$-harmonic)

$$\begin{aligned}\frac{\partial^2}{\partial t \partial s} E_p(u_{s,t}) = & \int_M (p-2)|du_{s,t}|^{p-4} \big(\sum_i < \nabla^F_{e_i} V, F_*(e_i) > \big) \\ & \times \big(\sum_j < \nabla^F_{e_j} W, F_*(e_j) > \big) dx \\ & + \int_M |du_{s,t}|^{p-2} \big(\sum_{i=1}^n < \nabla^F_{e_i} \nabla^F_{\frac{\partial}{\partial t}} W + R^N(W, F_*(e_i))V, F_*(e_i) > \\ & + \sum_{i=1}^n < \nabla^F_{e_i} V, \nabla^F_{e_i} W > \big) dx\end{aligned}$$

where $W = F_*(\frac{\partial}{\partial s})$ and $R^N$ is the curvature tensor on $N$ and $< R^N(x,y)y, x >$ denotes the sectional curvature of $N$ for the plane spanned by its orthonormal basis $\{x, y\}$

**2.8. Corollary.** *Suppose either for each fixed $x \in M$, the curve $F(x,t)$ is a constant speed geodesic in $M$ or $u$ is a p-harmonic map with compactly supported $V(x,0)$ in the interior of $M$. Then*

$$\begin{aligned}\frac{d^2}{dt^2} E_p(u_t)|_{t=0} = & \int_M (p-2)|du|^{p-4} \big(\sum_i < \nabla^u_{e_i} v, \widetilde{e}_i > \big)^2 \\ & + |du|^{p-2} \big(|\nabla^u v|^2 + \sum_i R^N(v, \widetilde{e}_i)v, \widetilde{e}_i > dx\end{aligned}$$

*where $\widetilde{e}_i$ means $du(e_i)$.*

As an immediate consequence of Corollary 2.7,

**2.9. Corollary.** *Let the sectional curvature of $N$ be nonpositive. Suppose either of the following two conditions holds:*

(i) *$F : M \times [0,1] \to N$ be a homotopy of $u$ through p-harmonic maps or*
(ii) *$F : M \times [0,1] \to N$ be a geodesic homotopy of $u$ ( i.e. for each $x \in M, F(x,t)$ is a geodesic, parametrized by constant speed for $t \in [0,1]$).*

*Then the function $t \mapsto E_p(u_t)$ is a convex function on $[0,1]$ for $p \geq 2$.*

Let $\triangle$ be the Hodge-DeRham Laplacian in which $\triangle = -(dd^* + d^*d)$. $Ric^M$ and $Riem^N$ denote the Ricci curvature of $M$ and the sectional curvature of $N$ respectively.

**2.10. The Bochner Formula.** ([75], (19)).

$$\begin{aligned}\frac{1}{2}\triangle |du|^{2\beta} &= \beta|du|^{2\beta-2}(\langle \triangle du, du\rangle + |\nabla du|^2 \\ &- \sum_{ij}\langle R^N(du(e_i), du(e_j))du(e_j), du(e_i)\rangle \\ &+ \sum_i \langle du(Ric^M e_i), du(e_i)\rangle + 2(\beta-1)|\nabla |du||^2)\end{aligned}$$

*at all points where* $|du| \neq 0$

This formula is globally true in the distribution sense, even if $|du|$ vanishes at various points. As an application of this Bochner formula, we have

**2.11. Theorem.** *Let* $u : M \to N$ *be a p-harmonic map in which* $p \geq 2$ *and suppose* $Ric^M \geq 0$ *and* $Riem^N \leq 0$. *Then*

(1) $u$ *is constant or totally geodesic*
(2) *If* $Ric^M > 0$ *at a point, then* $u$ *is constant.*
(3) *If* $Riem^N < 0$, *then* $u$ *is either constant or of rank one, in which case its image is a closed geodesic.*

## 3. An Extrinsic Average Variational Method.

In dealing with $p$-harmonic maps from or into $n$-manifolds with positive Ricci curvature and $n < p$ where well-known analytic tools such as heat flow method and Sobolev embedding theorem do not seem to yield much information, one may consider a geometric approach - an extrinsic average variational method [65]. Let $M \hookrightarrow R^q$ be an isometric immersion and $\{V_1^T, \cdots, V_q^T\}$ be the tangential projection of an orthonormal frame $V_1, \cdots, V_q$ in $R^q$ onto $M$. Denote $\phi_t^{V_j^T}$ by the flow generated by $V_j^T$. Let $u_t = u \circ \phi_t^{V_j^T}$, $A^{\nu_\alpha}$ be the Weingartan map and $Q^M$ be as in (3.4) where $N = M$. We obtain

**3.1. An Average Second Variational Formula on The Domain For The $p$-energy of A $p$-harmonic Map $u : M^n \to N^k$.**

$$\sum_{j=1}^{q} \frac{d^2}{dt^2} E_p(u \circ \phi_t^{V_j^T}) = \int_M |du|^{p-4}\{(p-2)\sum_{\alpha=1}^{q-n}\Big(\sum_{i=1}^{n} < \widetilde{A^{\nu_\alpha}(e_i)}, \tilde{e}_i >\Big)^2$$
$$+ |du|^2 \sum_{i=1}^{n} < \widetilde{Q^M(e_i)}, \tilde{e}_i >_N\}dx$$

where $\tilde{}$ means $du(\cdot)$, $\{e_1, \cdots, e_n\}$ is tangent to $M$ and $\{\nu_1, \cdots, \nu_{q-n}\}$ is normal to $M$.

Similarly, we can isometrically immerse $N^k$ into $R^q$ and choose an adopted orthonormal basis $\{V_1, \cdots, V_q\}$ in $R^q$ such that $\{V_1, \cdots, V_k\}$ is tangent to $N^k$. Then apply $u_i = \phi_t^{V_j^T}$, $u_0 = u, s = t$ in the second variational formula 2.7.

**3.2. An Average 2nd variational formula on the target for $p$-energy of $u : M^n \to N^k$** ($u$ is not necessarily $p$-harmonic).

$$\sum_{i=1}^{q} \frac{d^2}{dt^2}|_{t=0} E_p(\phi_t^{V_j^T} \circ u) = \int_M |du|^{p-4}\{(p-2)|\sum_{i=1}^{n} h(\widetilde{e}_i, \widetilde{e}_i)|^2$$
$$+ |du|^2 \sum_{i=1}^{n} < Q^N(\widetilde{e}_i), \widetilde{e}_i >_N\} dx$$

Via this average process in the Calculus of Variations and a linearization in the integrands in formulas (3.1) and (3.2), we find a large class of manifolds with positive Ricci curvature:

**3.3. Definition.** A Riemannian manifold $N$ is said to be **superstrongly unstable** (SSU) , if there exists an isometric immersion in $R^q$ such that, for every *unit* tangent vector $X$ to $N$ at every point $y \in N$, the following symmetric linear operator $Q_y^N$ is *negative definite.*

$$\langle Q_y^N(X), X\rangle_N = \sum_{i=1}^{k}(2|h_{X,\alpha_i}|^2 - h_{X,X} \cdot h_{\alpha_i,\alpha_i}) \tag{3.4}$$

and $N$ is said to be **p-superstrongly unstable** ($p$-SSU) for $p \geq 2$ if the following functional is *negative valued.*

$$F_{p,y}(X) = (p-2)|h_{X,X}|^2 + \sum_{i=1}^{k}(2|h_{X,\alpha_i}|^2 - h_{X,X} \cdot h_{\alpha_i,\alpha_i}) \tag{3.5}$$

where $h$ is the second fundamental form of $N$ in $R^q$ with standard inner product $v \cdot w$ and Euclidean norm $|v|$ defined for $v, w$ in $R^q$; and $\{\alpha_1, \ldots, \alpha_k\}$ is an orthonormal frame on $N$.

**3.6. Example.** The simplest example of an SSU manifold is $S^k$ where $k > 2$ in which $\langle Q_y^N(X), X\rangle_N = 2 - k$ and the simplest example of a $p$-SSU manifold is $S^k$ where $k > p$ in which $F_{p,y}(X) = p - k$.

**3.7. Definition.** A **p-SSU index** $w_p$ on a $p$-SSU manifold $N$ is defined to be

$$w_p = \inf_{y \in N} \varphi_p(y)/(k + p - 2)S(y)$$

where $\varphi_p(y) = \inf_{|x|=1}(-F_{p,y}(X))$ and $S(y) > 0$ be a positive upper bound of the sectional curvature of $N$ at $y$.

**3.8. Example.** The simplest $p$-SSU index $w_p$ on a $p$-SSU manifold $N$ is $w_p = \frac{k-p}{k+p-2}$ on $S^k$ for $k > p$ in which $\varphi_p(y) = k - p$ and $S(y) = 1$

A $p$-SSU index $w_p$ plays a dominating role in the regularity of a $p$-minimizer $u$ into a $p$-SSU manifold. In fact, via the Gauss curvature equation, (3.5) implies that on a $p$-SSU manifold $N$

$$-(k+p-2)S(y) < F_{p,y}(X) \leq -\varphi_p(y)$$

everywhere and hence, $w_p < 1$. However, (as indicated quantitatively from §4) the closer this $p$-SSU index $w_p$ is to 1, the easier $u$ is to establish the Liouville theorem (in terms of the volume growth condition in the domain), and the "smoother" the map is (in terms of the Hausdorff dimension of the singular set in the domain).

## 4. Further Regularity Theorems.

Applying Theorem 1.3, we refine Theorem 1.4 in [75] and [67]:

**(i) Maps into manifolds with nonpositive sectional curvature or a domain of a strictly convex function.**

**4.1. Theorem.** *Every p-minimizing, $L_1^p$ map into a manifold of non-positive sectional curvature, or into the domain of a strictly convex function, is $C^{1,\alpha}$. In particular, every p-minimizing $L_1^p$ map into either a complete noncompact manifold with positive sectional curvature or an open hemisphere is $C^{1,\alpha}$.*

**(ii) Maps into manifolds of positive Ricci curvature.**

For each constant $0 < \sigma \leq 1$, and each $p$-SSU index $w_p$ on a $p$-SSU manifold $N$, we set an integer

$$d(w_p, \sigma) = \begin{cases} 2 + [[p + 2\sqrt{p-\sigma}]] & \text{if} \quad w_p \geq \frac{p+1-\sigma+2\sqrt{p-\sigma}}{p+1+2\sqrt{p-\sigma}} \\ \max 1 + [\frac{\sigma}{1-w_p}], [p] & \text{otherwise} \end{cases} \tag{4.2}$$

where

$[[t]]$, the greatest integer [t] of $t$, if $t$ is not an integer; otherwise $[[t]] = t - 1$. (4.3)

**4.4. Regularity Theorem.** *Let $u \in L_1^p(M, N)$ be p-minimizing, and $u(x) \in N_0$, a.e., for a compact subset $N_0$ of p-SSU manifold $N$ with p-SSU index $w_p$. Then $u$ is locally Hölder continuous up to the boundary and the gradient of $u$ is also locally Hölder continuous outside a closed singular set $S_p$ in the interior of $M$. Furthermore,* **for any fixed constant** $0 < \sigma \leq 1$,

$$\dim(S_p) \leq \begin{cases} n - 3 - [[p + 2\sqrt{p-\sigma}]] & \textit{if} \quad w_p \geq \frac{p+1-\sigma+2\sqrt{p-\sigma}}{p+1+2\sqrt{p-\sigma}} \\ \max\{n - 2 - [\frac{\sigma}{1-w_p}], n - [p] - 1\} & \textit{otherwise} \end{cases}$$

**4.5. Corollary.** *If $u \in L_1^p(M, S^k)$ is p-minimizing, then*

$$\dim(S_p) \le \begin{cases} n-3-[[p+2\sqrt{p-\sigma}]] & \text{if} \quad (2p^2-\sigma p-2(1-\sigma)+4(p-1)\sqrt{p-\sigma})/\sigma \le k \\ \max\{n-2-[\frac{\sigma(k+p-2)}{2p-2}], n-[p]-1\} & \\ \qquad \text{if} \quad p < k < (2p^2-\sigma p-2(1-\sigma)+4(p-1)\sqrt{p-\sigma})/\sigma \end{cases}$$

*for any constant* $0 < \sigma \le 1$.

**(iii) Maps into manifolds with boundary.**

Let

$$c_{\sigma,p} = \frac{p+1-\sigma+2\sqrt{p-\sigma}}{p+1+2\sqrt{p-\sigma}} \tag{4.6}$$

set

$$d(a,\sigma,p) = \begin{cases} 2+[p+2\sqrt{p-\sigma}] & \text{if} \quad \sqrt[3]{c_{\sigma,p}} \le a \le \frac{2+c_{\sigma,p}}{1+2c_{\sigma,p}}, \\ 1+[\frac{\sigma}{1-a^3}] & \text{if} \quad a < \sqrt[3]{c_{\sigma,p}}, \\ 1+[\frac{\sigma(2a-1)}{3a-3}] & \text{if} \quad \frac{2+c_{\sigma,p}}{1+2c_{\sigma,p}} < a < 2. \end{cases} \tag{4.7}$$

and let the closed upper half-ellipsoid

$$\overline{(E_a^k)}_+ = \{(x_1, \dots, x_{k+1}) \in R^{k+1} : ax_1^2 + x_2^2 + \cdots + x_{k+1}^2 = 1 \quad \text{and} \quad x_{k+1} \ge 0\}.$$

Then we have

**4.8. Theorem.** *If $u \in L_1^p(M^n, \overline{(E_a^k)}_+)$ is p-minimizing, then u is locally Hölder continuous up to boundary and the gradient of u is also locally Hölder continuous in the interior of M for $n < d(a,\sigma,p)+1$, has at most isolated singularities for $n = d(a,\sigma,p)+1$ and has a closed singular set of Hausdorff dimensions $\dim(S_p) \le n - d(a,\sigma,p) - 1$ for $n > d(a,\sigma,p)+1$ where σ is any fixed constant satisfying $0 \le \sigma \le 1$.*

The above estimate which augments our previous regularity result on an open upper-hemisphere $S_+^k$ is optimal in the sense that if $a = 1$, then $\sigma = 0$, $c_{\sigma,p} = 1$, $\overline{(E_a^k)}_+ = \overline{S}_+^k$ and the estimate on $\dim(S_p)$ is the best possible result because of the equator map $u(x) = \left(\frac{x}{|x|}, 0\right)$ from the unit ball in $R^n$ to $S^n$. As a consequence, we obtain immediately the following

**4.9. Theorem.** *Let $u: M^n \to \overline{S}_+^k$ be an $L_1^p$ map which minimizes p-energy on each compact domain of M. Then u is locally Hölder continuous up to the boundary and the gradient of u is also locally Hölder continuous outside a closed interior singular set $S_p$ of $\dim(S_p) \le n - 3 - [p + 2\sqrt{p}]$.*

This result (4.9) is also due to Frank Duzaar.

## 5. Existence Theorems.

### (i) Representing components of the space $C^0(M,N) \cap L_1^p(M,N)$ by $p$-harmonic maps

In [67] we use the direct method in the Calculus of Variations [2,72] and the regularity theory [30,75] to obtain an existence theorem for $p$-harmonic maps, generalizing the work of Eells-Sampson [19], Schoen-Yau [58] and Burstall [2] which treat the case $p = 2$.

**5.1. Theorem.** *Let $M$ be a complete Riemannian $n$-manifold and $N$ be a compact Riemannian manifold with a contractible universal cover $\widetilde{N}$ and assume that $N$ has no non-trivial $p$-minimizing tangent map of $R^\ell$ for $\ell \leq n$. Then any continuous (or more generally $L_1^p-$) map $u$ from $M$ into $N$ of finite $p$-energy can be deformed to a $C^{1,\alpha}$ $p$-harmonic map $u_0$ minimizing $p$-energy in the homotopic class, where $1 < p < \infty$.*

There are various classes of manifolds $N$ which satisfy the above condition by the results of Bishop-O'Neill, Eberlein, Burns and Sacks-Uhlenbeck ([5,,13,6,49]). In particular, we have

**5.2. Corollary.** *Let $M$ be a complete Riemannian $n$-manifold and $N$ be a compact Riemannian manifold with convex supporting universal cover $\widetilde{N}$. Then any homotopy class of continuous maps of $M$ into $N$ containing a map of finite $p$-energy contains a $C^{1,\alpha}$ $p$-harmonic map minimizing $p$-energy in the homotopic class.*

**5.3. Corollary** [67].*Let $N$ be a compact Riemannian manifold with nonpositive sectional curvature or, more generally, suppose the universal cover $\widetilde{N}$ has no focal point or let $N$ be a closed surface with no conjugate points. Then any continuous map on a complete Riemannian manifold $M$ to $N$ of finite $p$-energy is homotopic to a $C^{1,\alpha}$ $p$-minimizer.*

### (ii) Representing homotopy classes by $p$-harmonic maps of least $p$-energy.

Just as harmonic forms represent cohomology groups, stable minimal varieties represent homology groups, or geodesics represent fundamental groups, so do $p$-harmonic maps represent homotopy groups. This is one of the features that distinguishes $p$-harmonic maps from harmonic maps. For completeness, we prove the following two theorems in [68].

**5.4. Theorem** *If $M$ is a compact Riemannian manifold, then for any positive integer $k$, each class in $\pi_k(M)$ can be represented by a $C^{1,\alpha}$ $p$-harmonic map $u_0$ from $S^k$ into $M$ minimizing $p$-energy in its homotopy class for any $p > k$.*

*Proof.* Let $\phi : S^k \to M$ be a $C^1$ map for any $p > k$. Define $H_\phi = \{u \in L_1^p(S^k, M) :$ $u$ is homotopic to $\phi\}$. Since $u$ is $C^\beta$, where $\beta = 1 - \frac{k}{p}$ by the Sobolev embedding theorem, $H_\phi$ is well-defined and nonempty.

Denote $I_p = \inf\{E_p(u) : u \in H_\phi\}$. Then $I_p < \infty$ since $E_p(\phi) < \infty$. Suppose $\{u_j\}$ is a sequence in $H_\phi$ such that $\lim_{j\to\infty} E_p(u_j) = I_p$. Since $M$ is compact, $\{u_j\}$ is an $L_1^p$-bounded subset. The Sobolev embedding theorem then implies that $\{u_j\}$ is a relative compact subset in $C^\gamma$ norm, for all $0 < \gamma < 1 - \frac{k}{p}$ and hence the limits $u_0$ of some convergent subsequence, denoted by $\{u_j\}$ again is $C^\gamma$. In particular, $u_0$ is the limit of a uniformly convergent subsequence $\{u_j\}$ in $C^0$ norm. Since $M$ is compact and any geodesic ball of sufficiently small radius is strongly convex, $u_0$ is homotopic to $u_j$ for sufficiently large $j$ and hence by the transitivity of homotopy, $u_0$ is in $H_\phi$. Therefore, $E_p(u_0) \geq I_p$. The lower semicontinuity of $E_p$ then implies that $E_p(u_0) = I$ and hence by Theorem 3.1 in [30], $u_0$ is $C^{1,\alpha}$. Thus, we have shown that every $C^1$ map $\phi: S^k \to M$ can be deformed to a $C^{1,\alpha}$ $p$-harmonic map $u_0$ minimizing $p$-energy in the homotopy class. In particular, $u_0$ is $p$-stable.

**(iii) Representing $m$-th homotopy classes by $m$-harmonic maps**

Generalizing Cartan's Theorem [8] (in which $m = 1$), and Sachs-Uhlenbeck's Theorem [49] (in which $m = 2$), we have

**5.5. Theorem.** *For any positive integer $m$, each nontrivial class in $\pi_m(K)$ can be represented by a sum of $C^{1,\alpha}$ $m$-harmonic maps $u_j : S^m \to K, j = 1, \ldots, s$, for some positive integer $s$.*

*Proof.* Let $\phi: S^m \to K$ be a $C^1$ map representing a nontrivial class in $\pi_m(K)$. Then $\phi$ has finite $m$-energy and hence the map $\overline{\phi}(x) = \phi\left(\frac{x}{|x|}\right)$ is a finite $m$-energy extension of $\phi$ to $B^{m+1}$. Minimizing $m$-energy in the homotopy class of $\overline{\phi}$ which agrees with $\phi$ on $\partial B^{m+1} = S^m$, one obtains a $p$-minimizer $u \in L_1^m(B^{m+1}, K)$ extending $\phi$. By the boundary regularity of Hardt and Lin [30], $u$ is $C^\alpha$ at the boundary $\partial B^{m+1}$ and has isolated singularities $x_1, \ldots, x_s$, $s \geq 1$ in $B^{m+1}$ otherwise $\pi_m(K) = 0$. By [30], blowing up $u$ at each $x_j$, $1 \leq j \leq s$ one obtains a nontrivial $m$-minimizing tangent map defined on $R^{m+1}$(c.f. Definition 3.1.), hence a $C^{1,\alpha}$ $m$-harmonic map $u_j : S^m \to K$ by the restriction.

However,

**5.6. Theorem.** *For any positive integer $k$, if $\pi_k(M)$ is nontrivial and $2 \leq p < k$, then there is no nonconstant $p$-stable smooth map $S^k \to M$ which represents a nontrivial class in $\pi_k(M)$.*

**5.7. Remark.** Theorem 5.6 still holds even if the hypothesis on nontrivial $\pi_k(M)$ is dropped (c.f. Theroem 9.7).

## 6. Uniqueness

We study the strong resemblance of $p$-harmonic maps to geodesics in terms of uniqueness properties. It is well-known that every geodesic on a compact manifold of nonpositive sectional curvature minimizes length in its homotopy class; hence it is stable. By the use of the second variational formula 2.7, we generalize this result to $p$-harmonic maps:

**6.1. Theorem.** *Let $M$ be compact with possibly nonempty boundary $\partial M$ and $N$ be compact with nonpositive sectional curvature. Then every p-harmonic map $u_0 : M \to N$ minimizes p-energy in its homotopy class (of maps which agree with $u_0$ on $\partial M$ if $\partial M$ is nonempty). In particular, every p-harmonic map $u_0 : M \to N$ is p-stable.*

By virtue of the existence result 7.2 and the second variational formula 2.7, we obtain the following uniqueness results, generalizing the work of Hartman [28].

**6.2. Theorem.** *If $u_0$ and $u_1$ are homotopic p-harmonic maps from $M$ into $N$ with $Riem^N \leq 0$, then they are homotopic through p-harmonic maps $u_s(\cdot)$ and the p-energy is constant on any arcwise connected set of p-harmonic maps, i.e. $E_p(u_s) = E_p(u_0) = E_p(u_1)$ for $\forall s \in (0,1)$. Furthermore, each path $s \mapsto G(x,s)$ is a geodesic segment with length independent of $x \in M$.*

**6.3. Corollary.** *Let $Riem^N \leq 0$ and let $u_0, u_1 : M \to N$ be two p-harmonic maps. If $\partial M$ is non-void and $u_0 = u_1$ are homotopic relative to a Dirichlet problem, then $u_0 = u_1$.*

With the aid of Theorem 6.1, we have

**6.4. Corollary.** *Let $\partial M$ be empty, $Riem^N \leq 0$ and $u_0 : M \to N$ be a p-harmonic map. Assume that there is some point of $u_0(M)$ at which $Riem^N < 0$. Then $u_0$ is unique in its homotopy class unless it is constant or maps $M$ onto a closed geodesic $\sigma$ in $N$. In the latter case, we have uniqueness up to rotations of $\sigma$.*

**6.5. Corollary.** *If $\partial M$ is empty, $Riem^N < 0$, $u_0, u_1 : M \to N$ are homotopic p-harmonic maps and $u_0$ has rank greater than one somewhere, then $u_0 = u_1$.*

**6.6. Corollary.** *If $Riem^N \leq 0$ and two homotopic p-harmonic maps $u_0, u_1 : M \to N$ agree at one point, then $u_0 = u_1$.*

**6.7. Remark.** The hypotheses on curvature can not be dropped. Ynging Lee and Derchyi Wu [38] have counter-examples. In fact, given any tw compact surfaces $M$ and $N$ with the same genus $g > 1$, Y.I. Lee and D.C. Wu have constructed, in every homotopy class of diffeomorphic maps, there are at least two distinct harmonic maps between $M$ and $N$ with some appropriate metrics.

## 7. Dirichelet Problem.

By minimizing $p$-energy in a class $H_\phi$ of maps with fixed trace and modifying [2], [72] and [67] (as in the proof of Theorem 2.2), we solve in [68], the Dirichlet problem for $p$-harmonic maps to which the solution is due to Hamilton [26] in the case $p = 2$ and $Riem^N \leq 0$:

**7.1. Theorem.** *Let $M$ be a compact Riemannian $n$-manifold with boundary $\partial M$ and $N$ be a compact Riemannian manifold with a contractible universal cover $\widetilde{N}$. Assume that $N$ has no non-trivial $p$-minimizing tangent map of $R^\ell$ for $\ell \leq n$. Then any $u \in Lip(\partial M, N) \cap C^0(M, N)$ of finite $p$-energy can be deformed to a $p$-harmonic map $u_0 \in C^{1,\alpha}(M - \partial M, N) \cap C^\alpha(M, N)$ minimizing $p$-energy in the homotopic class with trace ( in the Sobolev space ) $u_0|_{\partial M} = u|_{\partial M}$ , where $1 < p < \infty$. In particular, every $u \in C^1(M, N)$ can be deformed to a $C^{1,\alpha}$ $p$-harmonic map $u_0$ in $M - \partial M$ minimizing $p$-energy in the homotopic class with Hölder continuous $u_0|_{\partial M} = u|_{\partial M}$.*

**7.2. Corollary.** *Let $M$ be a compact manifold with (possibly empty) boundary and $N$ as above. If $u \in C^1(M, N)$ is not $p$-harmonic, then $u$ can be deformed to a $C^{1,\alpha}$ $p$-harmonic map $u_0$ with $u_0|_{\partial M} = u|_{\partial M}$ and $E_p(u_0) < E_p(u)$.*

Let $\phi : \partial M \to N$ be a $C^1$ map, $C^0$-extendible to $M$ and denote by $C_\phi(M, N)$ the space of extensions of $\phi$ to $C^0$ maps $M \to N$. If $Riem^N \leq 0$, then one can combine the existence theorem 7.2 and uniqueness theorem 6.2 to obtain the following:

**7.3. Theorem of Existence and Uniqueness.** *If $\phi \in C^\alpha(\partial M, N)$, then every component of $C_\phi(M, N)$ with an element of finite $p$-energy contains a unique $p$-harmonic representative, which is $C^{1,\alpha}$ in $M - \partial M$, Hölder continuous up to $\partial M$ and an $E_p$-minimum.*

**7.4. Theorem.** *Let $M$ be a compact Riemannian $n$-manifold with boundary $\partial M$ and $N$ be a compact $p$-SSU manifold Then for any $\eta \in L^{1-\frac{1}{p},p}(\partial M, N)$ one can find a $p$-harmonic map $u_0 \in L^{1,p}(M, N)$ minimizing $p$-energy in the class of maps $u \in L^{1,p}(M, N)$ with trace $u|_{\partial M} = \eta$ , where $1 < p < \infty$. The regularity properties of $u_0$ are described in Theorem 3.1. In particular, every $u \in C^1(M, N)$ can be deformed to a $C^{1,\alpha}$ $p$-harmonic map $u_0$ in $M - \partial M - S_p$ minimizing $p$-energy in the homotopic class with Hölder continuous $u_0|_{\partial M} = u|_{\partial M}$, where $\dim(S_p) \leq d(w_p, \sigma)$ as described in (4.2).*

## 8. The $E_p$-Hessian, $p$-index and $p$-nullity of $Id$

In [68], we compute the $E_p$-Hessian $H^{E_p}_{Id}(v, w)$ of the identity map $Id$ on $C(Id^{-1}T(M))$ and the associated quadratic form $\varphi(v)$ in terms of the Jacobi operator $J_{Id}$ of the energy functional ($p = 2$) and the Lie derivative $L_v g$ of $g$ in the direction of $v$, generalizing the work of Yano [7] for $p = 2$.

**8.1. Theorem.**

$$\begin{aligned}\varphi(v) &= n^{\frac{p-2}{2}} \int_M \frac{p-2}{n}(\operatorname{div} v)^2 + \langle J_{Id}(v), v\rangle dx \\ &= n^{\frac{p-2}{2}} \int_M \frac{p-2-n}{n}(\operatorname{div} v)^2 + \frac{1}{2}|L_v g|^2 dx.\end{aligned}$$

As an application, we link $p$-stability to eigenvalues and scalar curvature.

**8.2. Theorem.** *Suppose that $M$ is a compact Einstein manifold. $Id_M$ is p-unstable iff $\lambda_1 < \frac{2Scal^M}{n+p-2}$, where $\lambda_1$ is the first positive eigenvalue of $\Delta$ on functions.*

The bound on the eigenvalue is sharp (c.f. Remark 5.3). The case $p = 2$ is due to E. Mazet [40] and R. T. Smith [53] (c.f. [15]) and the case $p = 2$ in Theorem 8.4 is due to Urakawa ([64]). Let $\lambda(r) = \#\{\text{eigenvalues } \lambda \text{ of } -\Delta : 0 < \lambda < r\}$ and $m(r)$ be the multiplicity of $r$ (with $m(0)$ defined to be 0). Then by Theorem 8.2 we have

**8.3. Theorem.** *Let $M$ be a closed oriented Einstein manifold with $Ric^M = cg$ for some constant $c$. Then*

(a) $p - index(Id_M) = \lambda(\frac{2nc}{n+p-2})$
(b) $p - nullity(Id_M) = \dim(\underline{i}) + m(\frac{2nc}{n+p-2})$.

*where $\underline{i}$ denotes the algebra of infinitesimal isometries, i.e. of vector fields $v$ satisfying $L_v g = 0$.*

**8.4. Theorem.** *The identity map on every compact manifold of constant curvature is p-stable except for the standard unit sphere $S^n$. In particular, $Id_{S^n/\Gamma}$ is p-stable where $\Gamma \neq \{e\}$ is a finite group of isometries acting freely on $S^n$.*

**8.5. Theorem.** *The identity map on every compact manifold which supports a nonisometric, conformal vector field $v$ is p-unstable for $p < n$.*

**8.6. Theorem.** *Let $M$ be a compact manifold with $Ric^M \leq 0$. Then $Id_M$ is p-stable and $nullity(Id_M) \leq n$.*

## 9. Applications in Geometry, Topology and Analysis

Theorems 5.4 and 5.5 can be interpreted as

**9.1. A New Generalized Principle of Synge** *If there are no nonconstant $C^{1,\alpha}$ p-stable maps from $S^i$ into $M$ for some $p > i$, then $\pi_i(M) = 0$. Furthermore, if there are no nonconstant $C^{1,\alpha}$ m-harmonic maps from $S^m$ into $M$ for any $m > 1$, then $\pi_m(M) = 0$.*

This principle gives a new role in Riemannian geometry to $p$-harmonic maps, resulting in the new proofs of

**9.2. Cartan-Hadmard Theorem.** *Every compact Riemannian manifold $M$ with nonpositive sectional curvature is $K(\pi, 1)$.*

**9.3. Preissman Theorem.** *On a compact manifold $M$ with negative sectional curvature, every abelian subgroup of the fundamental group $\pi_1(M)$, different from the identity, is cyclic.*

**9.4. Gromoll-Wolf [25], Lawson-Yau [39] Theorem.** *Let $M$ be a compact manifold with nonpositive sectional curvature. Suppose the fundamental group $\pi_1(M)$*

*contains an Abelian subgroup of rank $k$. Then there exists an isometric immersion of a compact $k$-dimensional flat torus on $M$.*

**9.5. Bochner-Frankel Theorem.** *Let $M$ be a compact, orientable and $Riem^N \leq 0$, but not identically zero. Then its group of isometries is finite and no two elements are homotopic.*

The harmonic version of Theorems 9.4. and 9.5. are due to Hartman- Sampson [48] and Jost [33] respectively. the $p$-harmonic version of Theorem 9.4. is discussed in [74], and the case $p = 2$ in the following theorem is due to Eells-Sampson [14].

**9.6. Theorem.** *Let $N$ be a complete manifold with $Riem^N < 0$ and $N_0$ be a closed totally geodesic submanifold. Any nonconstant p-harmonic map of a compact manifold into a tubular neighborhood of $N_0$ has its image in $N_0$.*

Via an extrinsic average variational method, we obtain a nonexistence result

**9.7. Theorem.** *Every p-SSU manifold $N$ is p-SU; i.e. $N$ can neither be the domain nor the target of any nonconstant p-stable maps (into a complete manifold or from a compact manifold)(c.f. Theorem* 3.1.[75], *p*.251).

Balancing the above existence theorem (5.4. or Principle 9.1.) and nonexistence theorems, we have

**9.8. Theorem.** *Every compact p-SSU manifold $N$ is [p]-connected.*

In summary, we have

**9.8' Theorem.** *Let $N$ be a compact p-SSU manifold. Then the following assertions* (1) *through* (10) *hold:*

(1) $Ric^N > 0$.
(2) $\dim N > p$.
(3) *$N$ can not be the domain of any nonconstant p-stable map.*
(4) *$N$ can not be the target of any nonconstant p-stable map.*
(5) *The identity map $Id_N$ is p-unstable.*
(6) *$N$ is [p]-connected.*
(7) *$C^1(M, N)$ is dense in $L_1^p(M, N)$ for any compact manifold $M$ with possibly non-empty boundary.*
(8) $\inf\{E_p(\eta) : \eta$ *is homotopic to* $\zeta : M \to N\} = \inf\{E_p(\eta) : \eta$ *is homotopic to* $\xi : N \to K\} = 0$ *for any compact manifold $M$, any map $\zeta : M \to N$, any complete manifold $K$ and any map $\xi : N \to K$.*
(9) *$N$ is homeomorphic to k-sphere $S^k$ if $2 \leq p < k \leq 2p+1$ or $k = 3$, $p \geq 1$.*
(10) *Given any boundary data $\eta \in L^{1-\frac{1}{p},p}(\partial M, N)$ the Dirichelt problem for p-harmonic map has a solution $u_0$ ($\in L^{1,p}(M, N)$ minimizing p-energy in the class of maps $u \in L^{1,p}(M, N)$ with $u|_{\partial M} = \eta$ , where $1 < p < \infty$) of which regularity properties are described in Theorem* 4.4.

This leads to

**9.9. Topological Vanishing Theorem.** *Let $N$ be a minimal $k$-submanifold of a unit Euclidean sphere $S^{q-1}$ such that the Ricci curvature $Ric^N$ of $N$ satisfies*

$$Ric^N > k\left(1 - \frac{1}{p}\right) \tag{9.10}$$

*where $2 \le p < k$. Then $N$ is $p$-SSU. As a consequence, $\pi_1(N) = \cdots = \pi_{[p]}(N) = 0$. Furthermore, $C^1(M,N)$ is dense in $L_1^p(M,N)$, $N$ is $p$-SU and $\inf\{E_p(\eta) : \eta$ is homotopic to $\zeta : M \to N\} = \inf\{E_p(\eta) : \eta$ is homotopic to $\xi : N \to K\} = 0$ for any compact manifold $M$, any map $\zeta : M \to N$, any complete manifold $K$ and any map $\xi : N \to K$. Further, the Dirichlet problem for $p$-harmonic map is solvable for any boundary data $\eta \in L^{1-\frac{1}{p},p}(\partial M, N)$. In particular, if (9.10) holds for $2 \le p < k \le 2p+1$, or holds for $k = 3$, $p \ge 1$($N$ is not necessarily minimal), then $N$ is homeomorphic to $S^k$.*

**Proof.** For any unit vector $X \in T_p(N)$, choose an orthonormal frame $\{\alpha_1, \ldots, \alpha_k\}$ on $N$ such that at $p$, $\alpha_k = X$. For $1 \le i \le j \le k$, let $R_{ij}$ (resp. $\overline{R}_{ij}$) denote the sectional curvature of $N$ (resp. the unit sphere $S^{q-1}$) for the section $\alpha_i \wedge \alpha_j$. For $1 \le i, j \le k$, let $h_{ij} = (\nabla_{\alpha_i}\alpha_j)^{\perp}$ where $\nabla$ is the Riemannian connection on $S^{q-1}$ and $(\ \ )^{\perp}$ is the projection onto the normal space $T_p(N)^{\perp}$. The fact that $N$ is minimal says

$$\sum_{i=1}^{k} h_{ii} = 0 \tag{9.11}$$

Let $\overline{h}(Y,Z) = \langle \overline{\nabla}_Y Z, v\rangle$ where $\overline{\nabla}$ is the Riemannian connection on $R^q$, $Y$ and $Z$ are local vector fields on $S^{q-1}$ and $v$ is the unit normal vector to $S^{q-1}$. Then $\overline{h}(X,X) = 1$ and the second fundamental form $\tilde{h}$ of $N$ in $R^q$ splits into

$$|\tilde{h}(X,X)|^2 = h_{kk}^2 + |\overline{h}(X,X)|^2 \tag{9.12}$$
$$= h_{kk}^2 + 1. \tag{9.13}$$

Applying the Gauss-curvature equation, we have for $1 \le i \le k-1$

$$R_{ik} = \overline{R}_{ik} + \langle h_{ii}, h_{kk}\rangle - h_{ik}^2 \tag{9.14}$$

where $\langle\ ,\ \rangle$ is the Riemannian metric on $S^{q-1}$.

Summing (9.13) over $1 \le i \le k-1$ and applying (9.11), we have

$$Ric(X) = Ric(\alpha_k) = (k-1) - \sum_{i=1}^{k} h_{ik}^2 \tag{9.15}$$

In view of (9.14), the assumption (9.10) implies $\sum_{i=1}^{k} h_{ik}^2 < \frac{k-p}{p}$ and hence (9.15) $h_{kk}^2 < \frac{k-p}{p}$. On the other hand, (2.12) in ([32] p. 322) implies

$$\langle Q_y^N(X), X\rangle_N = \langle -2RicX + kX, X\rangle \tag{9.16}$$

Applying (3.5), (9.12), (9.15) we have under the assumption (9.10)

$$\begin{aligned} F_{p,y}(X) &= (p-2)(h_{kk}^2+1) + \langle Q_y^N(X), X\rangle_N \\ &< (p-2)(\frac{k-p}{p}+1) - 2(1-\frac{1}{p})k + k \\ &= 0\,. \end{aligned}$$

Hence $N$ is $p$-SSU and the first assertion now follows from Theorem 9.8.

If $2 \le p < k \le 2p+1$, then $N$ is homeomorphic to $S^k$ for $k \ge 4$ by Hurewicz theorem, the Poincaré duality Theorem, Smale's and Freedman's Theorems. If $k=3$ with weaker conditions that $1 \le p$ and $N$ is not necessary a minimal submanifold, then the metric on a compact 3-manifold $N$ with $Ric^N > 0$ , by a theorem of Hamilton ([27]), can be deformed to a metric $\tilde{g}$ with $Ric^{(N,\tilde{g})} \equiv c > 0$. But on a 3-manifold $N$, this is equivalent to the sectional curvature $Riem^{(N,\tilde{g})} \equiv \frac{c}{2} > 0$. It follows from the Cartan-Ambrose-Hicks theorem ([??]) that $N$ is homeomorphic to $S^3$. This completes the second assertion.

The conditions in Theorem 9.9. are sharp (c.f. [75], Remark on p.251-252); that is, there exist Clifford embeddings $N^k = S^p\left(\frac{1}{\sqrt{2}}\right) \times S^p\left(\frac{1}{\sqrt{2}}\right) \subset S^{2p+1}(1)$ with $Ric^N = k(1-\frac{1}{p})$ but $\pi_p(N) \ne 0$ for all $p$. There also exist totally geodesic embeddings $N^k = S^p(1)$ in the unit sphere with $Ric^N = k(1-\frac{1}{p})$ where $p=k$ but $\pi_p(N) \ne 0$. Furthermore, the Ricci curvature condition (9.10) is vacuous if $p$ is precisely $k$ by the following:

**9.17. Proposition.** *Let $N$ be a $k$-dimensional minimal submanifold of the unit Euclidean sphere $S^{q-1}(1)$ with Ricci curvature $Ric^N$. Then*

$$Ric^N \le k-1$$

*and equality holds if and only if $N$ is a totally geodesic submanifold of $S^{q-1}(1)$.*

As every compact, irreducible homogeneous space can be isometrically, minimally immersed into a Euclidean sphere [60], Theorem 9.9 implies

**9.18. Theorem.** *Let $N$ be a compact $k$-dimensional irreducible homogeneous space with the smallest positive eigenvalue $\lambda_1$ of $\Delta$ on functions. Then if (a) $\lambda_1 < \frac{p}{p-1}\frac{Scal^N}{k}$ is true, then the following assertions (b) through (i) hold:*

(b) *Every $p$-stable map $\varphi: M \to N$ from a compact manifold $M$ is constant.*
(c) *Every $p$-stable map $\psi: N \to K$ into a complete manifold $K$ is constant.*
(d) *The identity map $Id_N$ is $p$-unstable.*
(e) $\lambda_1 < \frac{2Scal^N}{k+p-2}$.
(f) $\pi_1(N) = \cdots = \pi_{[p]}(N) = 0$.
(g) $\inf\{E_p(\eta) : \eta$ *is homotopic to* $\zeta : M \to N\} = \inf\{E_p(\eta) : \eta$ *is homotopic to* $\xi : N \to K\} = 0$ *for any compact manifold $M$, any map $\zeta : M \to N$, any complete manifold $K$ and any map $\xi : N \to K$.*

(h) $C^1(M,N)$ *is dense in* $L_1^p(M,N)$.
(i) *Given any* $\eta \in L^{1-\frac{1}{p},p}(\partial M, N)$, *there is a p-harmonic map* $u_0 \in L^{1,p}(M,N)$ *minimizing p-energy in the class of maps* $u \in L^{1,p}(M,N)$ *with* $u|_{\partial M} = \eta$ , *where* $1 < p < \infty$. *The regularity properties of* $u_0$ *are described in Theorem* 4.4.

The case $p = 2$, as distinct from the case $p > 2$, $\frac{p}{p-1}\frac{Scal^N}{k} = \frac{2Scal^N}{k+p-2}$ i.e. $(a) \equiv (e)$. Hence with the same assumption on $N$ as in Theorem 9.12 we recover

**9.19. Theorem.** ([32],[43]) *The following properties are equivalent:*

($a'$) $\lambda_1 < \frac{2Scal^N}{k}$.
($b'$) *Every stable harmonic map* $\varphi: M \to N$ *from a compact manifold* $M$ *is constant.*
($c'$) *Every stable harmonic map* $\psi: N \to K$ *is constant.*
($d'$) *The identity map* $Id_N$ *is an unstable harmonic map.*

*In addition, any of the statements above implies*

($f'$) $\pi_1(N) = \pi_2(N) = 0$.
($g'$) $\inf\{E(\eta) : \eta$ *is homotopic to* $\zeta : M \to N\} = \inf\{E(\eta) : \eta$ *is homotopic to* $\xi : N \to K\} = 0$ *for any compact manifold* $M$, *any map* $\zeta : M \to N$, *any complete manifold* $K$ *and any map* $\xi : N \to K$.

The above results indicate a "gap phenomenon" when the first eigenvalue $\lambda_1$ is in $\left[\frac{p}{p-1}\frac{Scal^N}{k}, \frac{2Scal^N}{k+p-2}\right)$ for the case $p > 2$ and this gap is "filled in" when $p = 2$. Furthermore, this phenomenon dashes the hope that assertion $(d)$, the identity map on a compact irreducible homogeneous space $N$ is $p$-unstable, is strong enough to conclude assertions $(b)$ and $(c)$ that every nonconstant map from $N$ or to $N$ is $p$-unstable. We furnish an

**9.20. Example of a gap phenomenon.** The "gap" on the Cayley Plane is

$$\left[\frac{p}{p-1}\frac{Scal^N}{k}, \frac{2Scal^N}{k+p-2}\right) = \left[\frac{p}{2(p-1)}, \frac{16}{p+14}\right) \neq \emptyset \quad \text{for} \quad p \in [4, 10)\,.$$

Thus, for any $p \in (8, 10)$, the identity map on the Cayley plane is $p$-unstable by theorem 8.2. However, for the same range of $p$, the Cayley plane is the target of a $p$-stable map $: S^8 \to F_4/Spin(9)$ by Theorem 5.4, since $\pi_8(F_4/Spin(9)) \neq 0$. In sharp contrast, the identity map on Cayley plane is (2-)unstable and any nonconstant map into or from the Cayley plane is (2-)unstable ([32]).

As a further application of Theorem 9.9, we have

**9.21. Classification Theorem of compact, irreducible, p-SSU symmetric spaces.** *Let* $N$ *be a compact, irreducible p-SSU symmetric space. Then* $N$ *is one of the following*

(i) *the simply connected simple Lie groups* $(A_l)_{l\geq 1}$ *for* $p < 2 + \frac{2}{l^2+2l-1}$, $B_2 = C_2$ *for* $p < 2\frac{1}{2}$ *and* $(C_l)_{l\geq 3}$ *for* $p < 2 + \frac{1}{l}$

(ii) $SU(2n)/Sp(n)$, $n \geq 3$ *for* $p < 2 + \frac{n+1}{n^2-n-1}$
(iii) *Spheres* $S^k$ *for* $p < k$
(iv) *Quaternionic Grassmannians* $Sp(m+n)/Sp(m) \times Sp(n)$, $m \geq n \geq 1$ *for* $p < 2 + \frac{2}{m+n-1}$
(v) $E_6/F_4$, *for* $p < 3\frac{1}{4}$
(vi) *Cayley Plane* $F_4/Spin(9)$, *for* $p < 4$.

In the case $p = 2$, the above theorem recovers a theorem of Howard-Wei and Ohnita:

**9.22. Theorem([43][32]).** *Let N be a compact irreducible symmetric space. The following statements are equivalent :*

(a) *N is SSU.*
(b) *N is SU; i.e. N is neither the domain nor the target of any nonconstant stable harmonic map.*
(c) *N is U; i.e.* $Id_N$ *is an unstable harmonic map.*
(d) *N is one of the following:*

(i) *the simply connected simple Lie groups* $(A_l)_{l\geq 1}$, $B_2 = C_2$ *and* $(C_l)_{l\geq 3}$
(ii) $SU(2n)/Sp(n)$, $n \geq 3$
(iii) *Spheres* $S^k$, $k > 2$
(iv) *Quaternionic Grassmannians* $Sp(m+n)/Sp(m) \times Sp(n), m \geq n \geq 1$
(v) $E_6/F_4$
(vi) *Cayley Plane* $F_4/Spin(9)$.

The geometric measure theoretic approach yields the following topological information which extends 9.8′(6) and 9.8′(9):

**9.23. Theorem.** *Let N be a compact p-SSU manifold with a p-SSU index* $w_p$. *If* $k \leq 2d(w_p, \sigma) - 1$, *then N is homeomorphic to* $S^k$ *and if* $k > 2d(w_p, \sigma) - 1$, *N is* $(d(w_p, \sigma) - 1)$*-connected and each class in* $\pi_{d(w_p,\sigma)}(N)$ *is represented by a sum of* $C^{1,\alpha}$ $d(w_p, \sigma)$*-harmonic maps of* $S^{d(w_p,\sigma)} \to N$ *where* $d(w_p, \sigma)$ *is defined in* (4.2).

## References

[1] M. T. Anderson, The Bernstein problem in complete Riemannian manifolds, Berkeley Ph. D. Thesis, 1981.
[2] F. E. Burstall "Harmonic maps of finite energy from non-compact manifolds," *J. London Math. Soc.*, 30 (1984), 137-147.
[3] E. Bombieri, E. DeGiorgi and E. Giusti, "Minimal Cones and Bernstein Problem," *Invent. Math.*, 7 (1969), 243-268.
[4] F. Bethuel, "Approximation dans des espaces de Sobolev entre deux variétés et groupes d'homotopie," *C. R. Acad. Sc. Paris*, 307 (1988), 293-296.

[5] R. L. Bishop and B. O'Neill, "Manifolds of negative curvature," *Trans. Amer. Math. Soc.*, 145 (1969), 1-49.

[6] K. Burns, "Convex supporting domains on surfaces," *Bull. London Math. Soc.*, 17 (1985), 271-274.

[7] K. Yano and S. Bochner, "Curvature and Betti numbers," *Ann. of Math. Studies,* 32 (1953) Princeton University Press publaddr Princeton, N. J..

[8] E. Cartan, "Lecons sur la Geométrie des Espaces de Riemann," (1951) (2nd edition) Gauthier-Villars Paris.

[9] J. Douglas, "Solution of the problem of Plateau," *Trans. Amer. Math. Society,* 33 (1931) 263-321.

[10] E. De Giorgi, "Frontiere orientate di misura minima," *Sem. Mat. Sc. Norm. Sup. Pisa A. A.*, 8 (1960/61).

[11] L. C. Evans, "Partial regularity for stationary harmonic maps into spheres," *Arch. Rat. Mech. Anal.*, 7 (1983), 827-850.

[12] ——, "Estimates for smooth absolutely minimizing lipschitz extensions," *Electronic J. Diff. Equa.*, 03 (1993), 1-9.

[13] P. Eberlein, "When is geodesic flow of Anasov type II," *J. Differential Geom.*, 8 (1973), 565-577.

[14] J. Eells and L. Lemaire, "A Report on Harmonic Maps," *Bull. London Math. Sco.*, 10 (1978), 1-68.

[15] ——, "Selected Topics in Harmonic Maps," CBMS Regional Conf. Series Number, 50 (1983)

[16] ——, "Another Report on Harmonic Maps," *Bull. London Math. Sco.*, 20 (1988), 385-524.

[17] ——, "Some properties of Exponentially Harmonic Maps," PDEs, Banach Center Pub. 27 (1992), *Insti. of Math. Polish Acad. of Sci.* Warszawa, 129-136.

[18] J. Eells and A. Ratto, "Harmonic Maps and minimal immersions with symmetries," *Annals of Math. Studies,* 130 (1993), Princeton University Press.

[19] J. Eells and J. H. Sampson, "Harmonic mappings of Riemannian manifolds," *Amer. J. Math.*, 86 (1964), 109-160.

[20] H. Federer, *Geometric Measure Theory,* Springer-Verlag, N. Y. ,1969.

[21] ——, "The singular sets of rectifiable currents with codimension one and area minimizing flat chains modulo two with arbitrary codimensions," *Bull. A. M. S.*, 76 (1970).

[22] H. Federer and W. Fleming, "Normal and integral currents," *J. Diff. Geom.*, 17 (1982), 357-454.

[23] W. H. Fleming, "On the oriented Plateau problem," *Rend. Circolo Mat. Palermo,* 9 (1962), 69-89.

[24] M. Giaquinta and E. Giusti, "The singular set of the minimal of quadratic functionals," *Ann. Scuola Norm. Sup. Pisa Ser.*, IV 11 (1984), 45-55.

[25] D. Gromoll and J. Wolf, "Some relations between the metric structure and the algebraic structure of the fundamental group in manifolds of nonpositive curvature," *Bull. A. M. S.*, 77 (1971), 545-552.

[26] R. S. Hamilton, "Harmonic maps of manifolds with boundary," *Lecture Notes In Math,* No. 471 (1975), Springer, Berlin.

[27] — — R. S. Hamilton, "Three-manifolds with positive Ricci curvature," *Ann of Math.*, 72 (1960), 458-520.

[28] P. Hartman, "On homotopy harmonic maps," *Canad. J. Math.*, 19 (1967), 673-687.

[29] S. Hildebrandt, H. Kaul and K-O. Widman, "An existence theroem for harmonic mappings of Riemannian manifolds," *Acta Math.*, 138 (1977), 1-16.

[30] R. Hardt and F. H. Lin, "Mapping minimizing the $L^p$ norm of the gradient," *Common. Pure and Applied Math.*, XL (1987), 555-588.

[31] ——, "Regularity at infinity for area-minimizing hypersurfaces in hyperbolic space," *Invent. Math.*, 88 no.1 (1987), 217-224.

[32] R. Howard and S. W. Wei, "Non-existence of stable harmonic maps to and from certain homogeneous spaces and submanifolds of Euclidean space," *Trans. A. M. S.*, 294 no.1 (1986), 319-331.

[33] Jürgen Jost, "Nonlinear Methods in Riemannian and Kählerian Geometry," *DMV Seminar* Birkhaüser Verlag, 10 (1998).

[34] H. B. Lawson, "Minimal Varieties in real and complex geometry," *Les Presses De L'Univer sité De Montréal*, 1974.

[35] M. Li, "Partial regularity of energy minimizing harmonic maps into a complete manifold" (preprint).

[36] H. B. Lawson and J. Simons, "On stable currents, and their application to global problems in real and complex geometry," *Ann. of Math.*, 110 (1979), 127-142.

[37] S. Luckhaus, "Partial Hölder continuity for minima of certain energies among maps into a Riemannian manifold," *Indiana Univ. Math. J.*, 37 (1988), 349-367.

[38] Y. I. Lee and D. C. Wu, "A two dimensional non-rigidity Theorem."

[39] H. B. Lawson and S. T. Yau, "Compact manifolds of nonpositive curvature," *J. Differential Geom*, 7 (1972), 211-228.

[40] E. Mazet, "La formule de la variation seconde de l'energie au voisinage d'une application harmonique," *J. Differential Geom*, 8 (1973), 279-296.

[41] L. Mou and P. Yang, "Regularity for $n$-harmonic maps," *J. Geom. Anal.*, (to appear).

[42] S. Kobayashi and K. Nomizu, "Foundations of Differential Geometry," Interscience New York, II (1969).

[43] Y. Ohnita, "Stability of harmonic maps and standard minimal immersion," *Tohoku Math. J.*, 38 (1986), 259-267.

[44] R. S. Palais, "Critical point theory and the maximum principle," *Proc. Sympos. Pure Math.*, 15 (1970), 185-212.

[45] T. Rado, "On Plateau's problem," *Ann. of Math.*, 31 (1930), 457-469.

[46] P. H. Rabinowitz, "Variational methods for nonlinear eigenvalue problems," (*CIME, Verona,* 1974), *Ediz, Cremonese,* (1974), 141-195.

[47] R. Schoen, "Analytic aspects of the harmonic map problem," *Seminar on nonlinear partial differential equations,* Springer-Verlag, New York-Berlin-Heidelbert-Tokyo, 2 (1984), 321-358.

[48] J. H. Sampson, "Some properties and applications of harmonic mappings," *Ann. Ec. Norm. Sup.*, XI (1978).

[49] J. Sacks and K. Uhlenbeck, "The existence of minimal immersions of 2-spheres," *Ann. of Math.*, 113 (1981), 1-24.

[50] J. Schwartz, "Nonlinear functional analysis," Gordon and Breach, New York, 1969.

[51] J. Simons, "Minimal varieties in Riemannian manifolds," *Ann. of Math.*, 88 (1968), 62-105.

[52] S. Smale, "Generalized Poincaré's Conjecture in dimension greater than four," *Ann. of Math.*, 74 (1961), 391-406.

[53] R. T. Smith, "The second variational formula for harmonic mapping," *Proc. Amer. Math. Soc.*, 47 (1975), 229-236.

[54] E. Spanier, "Algebraic Topology," McGraw-Hill, 1966.
[55] J. L. Synge, "On the connectivity of spaces of positive curvature," *Quar. J. Math. Oxford,* 1 (1936), 316-320.
[56] R. Schoen and K. Uhlenbeck, "A regularity theory for harmonic maps," *J. Diff. Geom.,* 17 (1982), 307-335.
[57] ——, "Boundary regularity and the Dirichlet problem for harmonic maps" *J. Diff. Geom.,* 18 (1983), 253-268.
[58] R. Schoen and S. T. Yau, "Harmonic maps and the topology of stable hypersurfaces and manifolds of nonnegative Ricci curvature," *Comm. Math. Helv.,* 51 (1976), 333-341.
[59] ——, "Compact group actions and the topology of manifolds with nonpositive curvature," *Topology,* 18 (1979), 361-380.
[60] T. Takahashi, "Minimal immersions of Riemannian manifolds," *J. Math. Soc. Japan,* 18 (1966), 380-385.
[61] P. Tolksdorf, "Everywhere regularity for some quasi-linear systems with lack of ellipticity," *Annali di Math Pura et Applicata,* 134 (1983), 241-266.
[62] T. Toro and C. Y. Wang, "On completeness properties of weakly $p$-harmonic maps into homogeneous spaces," *Indiana Univ. Math. J.,* 44 (1995), 87-113.
[63] K. Uhlenbeck, "Regularity for a class of non-linear elliptic systems," *Acta Math.,* 138 (1977), 219-240.
[64] H. Urakawa, "Stability of harmonic maps and eigenvalues of the Laplacian," *Trans. Amer. Math. Soc.,* 301 (1987), 557-589.
[65] S. W. Wei, "An extrinsic average variational method," *Contemp. Math.,* 101 (1989), 44-78.
[66] ——, "Liouville theorem and regularity of minimizing harmonic maps into superstrongly unstable manifolds," *Contemp. math.,* 127 (1992), 131-154.
[67] ——, "The minima of the $p$-energy functional," *Elliptic and Parabolic Methods in Geometry* (1996), A. K. Peters, 171-203.
[68] ——, "Representing homotopy groups and spaces of maps by $p$-harmonic maps," (preprint).
[69] S. S. Wang, "Energy minimizing sections of a Riemannian fiber bundle," Ill. *J. of Math.,* (to appear) 31 (1996).
[70] ——, "$p$-Dirichlet energy minimizing maps into a complete manifold," (preprint).
[71] B. White, "Infima of energy functionals in homotopy classes of mapping," *J. Diff. Geometry,* 23 (1986), 127-142.
[72] ——, "Homotopy classes in Sobolev spaces and the existence of energy minimizing maps," *Acta Math.,* 160 (1988), 1-17.
[73] S. P. Wang and S. W. Wei, "Bernstein conjecture in hyperbolic geometry," *Seminar on Minimal Submanifolds,* edited by E. Bombieri, Annals of Math Studies, no.103 (1983), 339-358.
[74] S. W. Wei and R. L. Xia.
[75] S. W. Wei and C. M. Yau, "Regularity of $p$-energy minimizing maps and p-superstrongly unstable indices," *J. Geom. Analysis,* 4 (2) (1994), 247-272.

Department of Mathematics, University of Oklahoma, 601 Elm Avenue, Room 423, Norman, Oklahoma 73019-0315. U. S. A.
e-mail address: wwei@uoknor.edu