TAMKANG JOURNAL OF MATHEMATICS
Volume 28, Number 1, Spring 1997

A NOTE ON HADAMARD'S INEQUALITY

GOU-SHENG YANG AND MIN-CHUNG HONG

Abstract. In the present note we establish a new convex function related to the well known Hadamard's inequality by using a fairly elementary analysis.

1. Introduction

The following inequalities

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1}$$

which hold for all convex mappings $f : [a,b] \to R$, are known in the literature as Hadamard's inequalities [3]. We note that J. Hadamard was not first who discovered them. As is pointed out by D. S. Mitrinovic and I. B. Lackovic [4] the inequalities (1) are due to C. Hermite who obtained them in 1883, ten years before J. Hadamard.

In [2], S. S. Dragomir proved that there is a convex monotonically increasing function between $f(\frac{a+b}{2})$ and $\frac{1}{b-a}\int_a^b f(x)dx$. In this note, we shall establish that there is a convex monotonically increasing function between $\frac{1}{b-a}\int_a^b f(x)dx$ and $\frac{f(a)+f(b)}{2}$. As for other inequalities in connection with Hadamard's result see [1, 5, 6], and the references therein.

2. The Main Result

Now, for a given convex mapping $f : [a,b] \to R$, Let $F : [0,1] \to R$ be defined by

$$F(t) = \frac{1}{2(b-a)}\int_a^b \{f[(\frac{1+t}{2})a + (\frac{1-t}{2})x] + f[(\frac{1+t}{2})b + (\frac{1-t}{2})x\}dx. \tag{2}$$

The following theorem holds:

Theorem 1. *Let* $f[a,b] \to R$ *and* $F : [0,1] \to R$ *be as above. Then*

(i) *F is convex on* $[0,1]$,

Received August 23, 1995.
1991 *Mathematics Subject Classification.* 26D15.
Key words and phrases. Hadamard's inequality, convex.

(ii) *F is monotonically increasing on* $[0,1]$, *and*
(iii)

$$\inf_{t\in[0,1]} F(t) = F(0) = \frac{1}{b-a}\int_a^b f(x)dx,$$

$$\sup_{t\in[0,1]} F(t) = F(1) = \frac{f(a)+f(b)}{2}$$

Proof. (i) Let $\alpha,\beta \geq 0$ with $\alpha+\beta=1$ and $t_1,t_2\in[0,1]$. Then

$$\begin{aligned}
F(\alpha t_1+\beta t_2) &= \frac{1}{2(b-a)}\int_a^b \{f[\frac{1+(\alpha t_1+\beta t_2)}{2}a+\frac{1-(\alpha t_1+\beta t_2)}{2}x] \\
&\quad +f[\frac{1+(\alpha t_1+\beta t_2)}{2}b+\frac{1-(\alpha t_1+\beta t_2)}{2}x]\}dx \\
&= \frac{1}{2(b-a)}\int_a^b \{f[\alpha\frac{(1+t_1)a+(1-t_1)x}{2}+\beta\frac{(1+t_2)a+(1-t_2)x}{2} \\
&\quad +f[\alpha\frac{(1+t_1)b+(1-t_1)x}{2}+\beta\frac{(1+t_2)b+(1-t_2)x}{2}]\}dx \\
&\leq \frac{\alpha}{2(b-a)}\int_a^b \{f[\frac{(1+t_1)a+(1-t_1)x}{2}]+f[\frac{(1+t_1)b+(1-t_1)x}{2}]\}dx \\
&\quad +\frac{\beta}{2(b-a)}\int_a^b \{f[\frac{(1+t_2)a+(1-t_2)x}{2}]+f[\frac{(1-t_2)b+(1-t_2)x}{2}]\}dx \\
&= \alpha F(t_1)+\beta F(t_2),
\end{aligned}$$

so that F is convex on $[0,1]$.

(ii) Let $0\leq t<1$. Then

$$\begin{aligned}
F(t) &= \frac{1}{2(b-a)}\int_a^b \{f[\frac{(1+t)a+(1-t)x}{2}]+f[\frac{(1+t)b+(1-t)x}{2}]\}dx \\
&= \frac{1}{(1-t)(b-a)}[\int_a^{\frac{a+b}{2}-t(\frac{b-a}{2})} f(x)dx+\int_{\frac{a+b}{2}+t(\frac{b-a}{2})}^b f(x)dx].
\end{aligned}$$

Thus

$$\begin{aligned}
F'(t) &= \frac{1}{(1-t)^2(b-a)}[\int_a^{\frac{a+b}{2}-t(\frac{b-a}{2})} f(x)dx+\int_{\frac{a+b}{2}+t(\frac{b-a}{2})}^b f(x)dx] \\
&\quad +\frac{1}{(1-t)(b-a)}[f(\frac{a+b}{2}-t(\frac{b-a}{2}))(\frac{a-b}{2})-f(\frac{a+b}{2}+t(\frac{b-a}{2}))(\frac{b-a}{2})] \\
&= \frac{1}{(1-t)^2(b-a)}[\int_a^{\frac{a+b}{2}-t(\frac{b-a}{2})} f(x)dx+\int_{\frac{a+b}{2}+t(\frac{b-a}{2})}^b f(x)dx] \\
&\quad -\frac{1}{2(1-t)}[f(\frac{a+b}{2}-t(\frac{b-a}{2}))+f(\frac{a+b}{2}+t(\frac{b-a}{2}))]
\end{aligned}$$

$$= \frac{1}{(1-t)^2}\{\frac{1}{(b-a)}[\int_a^{\frac{a+b}{2}-t(\frac{b-a}{2})} f(x)dx + \int_{\frac{a+b}{2}+t(\frac{b-a}{2})}^b f(x)dx]$$
$$-\frac{(1-t)}{2}[f(\frac{a+b}{2}-t(\frac{b-a}{2}))+f(\frac{a+b}{2}+t(\frac{b-a}{2}))]\}$$

Let $G(t) = \frac{1}{b-a}[\int_a^{\frac{a+b}{2}-t(\frac{b-a}{2})} f(x)dx + \int_{\frac{a+b}{2}+t(\frac{b-a}{2})}^b f(x)dx] - \frac{(1-t)}{2}[f(\frac{a+b}{2} - t(\frac{b-a}{2})) + f(\frac{a+b}{2} + t(\frac{b-a}{2}))]$. Then

$$G'(t) = -\frac{1}{2}[f(\frac{a+b}{2}-t(\frac{b-a}{2}))+f(\frac{a+b}{2}+t(\frac{b-a}{2}))]+\frac{1}{2}[f(\frac{a+b}{2}-t(\frac{b-a}{2}))$$
$$+f(\frac{a+b}{2}+t(\frac{b-a}{2}))] - \frac{(1-t)(b-a)}{4}[f'(\frac{a+b}{2}+t(\frac{b-a}{2}))$$
$$-f'(\frac{a+b}{2}-t(\frac{b-a}{2}))]$$
$$= -\frac{t(1-t)(b-a)^2}{4}f''(c) \tag{3}$$

where c is a number between $\frac{a+b}{2} - t(\frac{b-a}{2})$ and $\frac{a+b}{2} + t(\frac{b-a}{2})$.

Since f is convex and $t \in [0,1)$, the last term of (3) is not greater than zero for all $t \in [0,1)$, so that G is monotonically decreasing on $[0,1]$.

Consequently, $F'(t) = \frac{1}{(1-t)^2}G(t) \geq \frac{1}{(1-t)^2}G(1^-) = \frac{1}{(1-t)^2}G(1) = 0$, for $0 \leq t < 1$ which shows that F is monotonically increasing on $[0,1)$.

(iii) We shall prove the following inequalities:

$$\frac{1}{b-a}\int_a^b f(x)dx \leq F(t) \leq \frac{(1+t)}{2}\frac{f(a)+f(b)}{2} + \frac{(1-t)}{2}\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{4}$$

for all t in $[0,1]$.

Because $F(t)$ is monotonically increasing, we have

$$F(t) \geq F(0) = \frac{1}{2(b-a)}\int_a^b [f(\frac{b+x}{2}) + f(\frac{a+x}{2})]dx = \frac{1}{b-a}\int_a^b f(x)dx$$

for all $t \in [0,1]$.

Now, using the convexity of f, we have

$$F(t) = \frac{1}{2(b-a)}\int_a^b \{f[(\frac{1+t}{2})a + (\frac{1-t}{2})x] + f[(\frac{1+t}{2})b + (\frac{1-t}{2})x]\}dx$$
$$\leq \frac{1}{2(b-a)}\int_a^b [(\frac{1+t}{2})(f(a)+f(b)) + (1-t)f(x)]dx$$
$$= \frac{(1+t)}{2}\frac{f(a)+f(b)}{2} + \frac{(1-t)}{2}\frac{1}{b-a}\int_a^b f(x)dx,$$

and the second inequality in (4) is proved.

Finally, by the Hadamard's inequalities, we have

$$\frac{(1+t)}{2}\frac{f(a)+f(b)}{2}+\frac{(1-t)}{2}\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{(1+t)}{2}\frac{f(a)+f(b)}{2}+\frac{(1-t)}{2}\frac{f(a)+f(b)}{2}$$
$$= \frac{f(a)+f(b)}{2}$$

This completes the proof.

Corollary. *Under the assumptions of Theorem 1, we have*

$$\frac{1}{2(b-a)}\int_a^b [f(\frac{3a+x}{4})+f(\frac{3b+x}{4})]dx - \frac{1}{b-a}\int_a^b f(x)dx$$
$$\leq \frac{f(a)+f(b)}{2} - \frac{1}{2(b-a)}\int_a^b [f(\frac{3a+x}{4})+f(\frac{3b+x}{4})]dx. \qquad (5)$$

Proof. Since F is convex on $[0,1]$, we have

$$\frac{1}{2(b-a)}\int_a^b [f(\frac{3a+x}{4})+f(\frac{3b+x}{4})]dx = F(\frac{1}{2})$$
$$= F(\frac{1}{2}\cdot 1+\frac{1}{2}\cdot 0)$$
$$\leq \frac{1}{2}F(1)+\frac{1}{2}F(0)$$
$$= \frac{1}{2}(\frac{f(a)+f(b)}{2})+\frac{1}{2}\frac{1}{b-a}\int_a^b f(x)dx.$$

Hence (5) follows immediately.

3. Applications

(1) Let $p \geq 1$, $0 \leq a < b$ and $f(x) = x^p$. Then

$$(\frac{a+b^p}{2}) \leq \frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)} \leq \frac{1}{(1-t)(p+1)(b-a)}\{[\frac{a+b}{2}-t(\frac{b-a}{2})]^{p+1}-a^{p+1}$$
$$+b^{p+1}-[\frac{a+b}{2}+t(\frac{b-a}{2})]^{p+1}\}$$
$$\leq \frac{a^p+b^p}{2}.$$

for all t in $[0,1)$.

(2) Let $0 < a < b$ and $f(x) = \frac{1}{\sqrt{x}}$, $x > 0$. Then

$$\frac{2}{\sqrt{a}+\sqrt{b}} \leq \frac{2}{(1-t)(b-a)}\{[\frac{a+b}{2}-t(\frac{b-a}{2})]^{\frac{1}{2}}-\sqrt{a}+\sqrt{b}-[\frac{a+b}{2}+t(\frac{b-a}{2})]^{\frac{1}{2}}\}$$
$$\leq \frac{1}{2}\frac{\sqrt{a}+\sqrt{b}}{\sqrt{ab}}$$

for all t in $[0,1)$.
(3) Let $0<a<b$ and $f(x)=\frac{1}{x}$, $x>0$. Then

$$\frac{\ln b-\ln a}{b-a} \le \frac{1}{(1-t)(b-a)} \ln \frac{b}{a} \frac{\left[\frac{a+b}{2}-t(\frac{b-a}{2})\right]}{\left[\frac{a+b}{2}+t(\frac{b-a}{2})\right]} \le \frac{a+b}{2ab}$$

for all $t \in [0,1)$.

Remark. The following inequalities can be found in [7, p.130],

$$\frac{1}{2}(a+b) > e^{-1}(\frac{b^b}{a^a})^{\frac{1}{b-a}} > \frac{b-a}{\ln b - lna} > \sqrt{ab} > ab\Big(\frac{\ln b-\ln a}{b-a}\Big) > e(\frac{a^b}{b^a})^{\frac{1}{b-a}} > \frac{2ab}{a+b}.$$

Consequently, our result from application (3) gives a refinement of this classic fact.

References

[1] H. Alzer, "A note on Hadamard's inequalities," *C. R. Math. Rep. Acad. Sci. Canada*, 11 (1989), 255-258.
[2] S. S Dragomir, "Two mappings in connection to Hadamard's inequalities," *J. Math. Anal. Appl.*, 167 (1992), 49-56.
[3] J. Hadamard, "Etude sar les proprietes des fonctions entieres et en particulier d'une fonction consideree par Riemann," *J. Math. Pure Appl.*, 58 (1893), 171-215.
[4] D.S Mitrinovic, and I. B. Lackovic, "Hermite and convexity," *Aequations Math.*, 28 (1985), 229-232.
[5] S. S. Dragomir, J. E. Percaric, and J. Sandor, "A note on the Jensen-Hadamard inequality," *Anal. Numer. Theor, Approx.*, 19 (1990), 29-34.
[6] J. Sandor, "Some integral inequalities," *Elem. Math.*, 43 (1988), 177-180.
[7] P. S. Bullen, D. S. Mitrinovic and P. M. Vasic (Eds.), *Means and Their Inequalities*. D. Reidel Publishing Company, 1988.

Department of Mathematics, Tamkang University, Tamsui, Taiwan, 25137.