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SUBCLASSES OF SPIRALLIKE FUNCTIONS DEFINED BY RUSCHEWEYH DERIVATIVES

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Abstract. An attempt is being made perhaps for the first time, to relate the spirallike functions with the concept of Ruscheweyh derivatives. A new subclass of spirallike functions is introduced with the help of Ruscheweyh derivatives.

1. Introduction

Let $\mathcal{A}$ denote the family of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

that are analytic in the unit disc $\Delta = \{z : |z| < 1\}$.

For f and g in $\mathcal{A}$, we say that f if subordinate to g denoted as $f \prec g$, if there exists a Schwarz's function $\omega(z)$ analytic in Δ with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g\{\omega(z)\}$.

Given complex numbers A, B with $|A| \leq 1$, $|B| \leq 1$ and $A \neq B$, a function f of $\mathcal{A}$ is said to be in $S^\alpha[A, B]$, where $|\alpha| < \pi/2$ if

$$e^{i\alpha} \frac{zf'(z)}{f(z)} \prec \cos\alpha \{\frac{1 + Az}{1 + Bz}\} + i \ \sin \ \alpha \tag{1.2}$$

for $z \in \Delta$.

This class was introduced and studied by Nikitin [3].

Let $K^\alpha[A, B]$, denote the class of all f in $\mathcal{A}$ satisfying the condition that $zf'(z)$ is in $S^\alpha[A, B]$. $S^\alpha[A, B]$ and $K^\alpha[A, B]$ are the subclasses of spirallike and Robertson functions respectively studied by several authors earlier. In particular $S^0[A, B] \equiv S[A, B]$, subclass of starlike functions and $K^0[A, B] \equiv K[A, B]$, subclass of convex functions K and $S^\alpha[1, -1]$ is the class of all $\alpha - \mathit{sprial}$ functions in Δ.

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Ruscheweyh [4], defined the symbol $D^n f(z)$, by

$$D^n f(z) = \frac{z(z^{n-1}f(z))^{(n)}}{n!}, \tag{1.3}$$

for $n \in N_0 = N \cup \{0\} = \{0, 1, 2, \cdots\}$. The symbol $D^n f$ is referred to as the n^{th} order Ruscheweyh derivative of f, for $f \in \mathcal{A}$

Given f of the form (1.1) we notice that

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z) = z + \sum_{k=2}^{\infty} \binom{k+n-1}{n} a_k z^k \tag{1.4}$$

for all z in Δ where the operator "$*$" is the usual Hadmard product [Convolution] of functions given by:

If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ then

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k \tag{1.5}$$

for all z in Δ.

We amalgamate the notions of Ruscheweyh and Nikitin to originate the class $H_n^\alpha[A, B]$. A function $f \in \mathcal{A}$ is said to be in $H_n^\alpha[A, B]$, if it satisfies the condition,

$$e^{i\alpha} z \frac{(D^n f)'}{(D^n f)} \prec \cos\alpha\{\frac{1+Az}{1+Bz}\} + i\sin\alpha \tag{1.6}$$

for all $z \in \Delta$, for some complex numbers A and B with $|A| \leq 1$, $|B| \leq 1$ and $A \neq B$. It is proved in Corollary 2 of Theorem 4 that the functions in $H_n^\alpha[A, B]$ are spirallike and hence are univalent in Δ.

Note that,

$$f \in H_n^\alpha[A, B] \text{ if and only if } D^n f = f * h_n \in S^\alpha[A, B],\ n \in N_0 \tag{1.7}$$

where

$$h_n(z) = \frac{z}{(1-z)^{n+1}} = z + \sum_{k=2}^{\infty} \binom{k+n-1}{n} z^k \tag{1.8}$$

If $n = 1$, then the relation (1.7) is equivalent to:

$$f \in K^\alpha[A, B] \text{ if and only if } zf'(z) \in S^\alpha[A, B].$$

By specializing the parameters in $H_n^\alpha[A, B]$, we obtain various subclasses studied by various authors earlier. We mention only few:

$H_n^0[A, B] = R_n[A, B]$, for A, B real numbers with $-1 \leq B < A \leq 1$ is the class introduced and studied by Ahuja in [1].

$H_0^\alpha[A, B] = S^\alpha[A, B]$, is the class introduced and studied by Nikitin in [3].

$H_1^\alpha[A,B] = K^\alpha[A,B]$, is studied in [2].

Furthermore we define the class $H_n^\alpha(a,b)$ as the class of functions f of $\mathcal{A}$ satisfying the condition

$$\left|\frac{\{e^{i\alpha}z(D^nf)'/D^nf\} - i\,\sin\alpha}{\cos\alpha} - a\right| < b \tag{1.9}$$

for all z in Δ, where $a \in C$, $b \in R$ with $Re\ a \geq b$ and $|a-1| < b$ and $|\alpha| < \pi/2$.

Though the family $H_n^\alpha(a,b)$ appears to be new, we have $H_0^\alpha(a,b) \equiv S^\alpha(a,b)$ and $H_1^\alpha(a,b) \equiv k^\alpha(a,b)$ were studied in [2]. Also $H_0^\alpha(\infty,\infty) \equiv S^\alpha(\infty,\infty)$ denotes the class of all $\alpha - spiral$ functions in Δ.

Finally note that for each $n \geq 0$, $H_n^\alpha[A,B] \subset H_n^\alpha[1,-1]$.

We conclude this part by giving a Lemma whose proof is omitted as it follows on the same lines as in Theorem 1 of [2].

Lemma 1.
(i) If A and B are complex numbers with $|A| \leq 1$, $|B| < 1$, $A \neq B$, $Im(A\overline{B}) = 0$, $n \in N$ and $|\alpha| < \pi/2$ then

$$H_n^\alpha[A,B] \equiv H_n^\alpha\left[\frac{1-A\overline{B}}{1-|B|^2}, \frac{|A-B|}{1-|B|^2}\right] \tag{1.10}$$

(ii) If $a \in C$, $b \in R$ with $Re\ a \geq b$ and $|a-1| < b$, $|\alpha| < \pi/2$ and $n \in N$ then

$$H_n^\alpha(a,b) \equiv H_n^\alpha\left[\frac{b^2 - |a|^2 + a}{b}, \frac{1-\overline{a}}{b}\right] \tag{1.11}$$

We denote the various subclasses with parenthesis if we are considering the functions in the disc formulation given in (1.9) and with brackets if we are considering subordination formulation given by (1.6).

2. Certain Characterizations

In this part, we shall assume that $f \in \mathcal{A}$ is of the form (1.1) with $X = \{\xi : |\xi| = 1\}$ and $\Delta_0 = \Delta - \{0\}$, is the punctured open disc.

Lemmm 2. [2] *The function f is in $S^\alpha[A,B]$, if and only if for all z in Δ_0, $\xi \in X$,*

$$f * \left\{\frac{z + Mz^2}{(1-z)^2}\right\} \neq 0 \tag{2.1}$$

where $M = \frac{e^{i\alpha} + (A\cos\alpha + iB\sin\alpha)\xi}{\cos\alpha(B-A)\xi}$.

Lemma 3. *The function f is in $H_n^\alpha[A,B]$ if and only if*

$$f * \left\{\frac{z - \left[\frac{(n+1)e^{i\alpha} + [(A+nB)\cos\alpha + (n+1)iB\sin\alpha]\xi}{\cos\alpha\ (A-B)\xi}\right]}{(1-z)^{n+2}}\right\} \neq 0 \tag{2.2}$$

for all $z \in \Delta_0$ and $\xi \in X$.

Proof. In view of the relation (1.7), an application of Lemma 2 exhibits that $f \in H_n^\alpha[A,B]$ if and only if

$$(f * h_n) * \left\{\frac{z+Mz^2}{(1-z)^2}\right\} \neq 0 \qquad (z \in \Delta_0, \xi \in X), \tag{2.3}$$

where

$$M = \frac{e^{i\alpha} + (A\cos\alpha + iB\sin\alpha)\xi}{\cos\alpha\,(B-A)\xi} \tag{2.4}$$

and h_n is as in (1.8).

Now after a brief calculation the left hand side of (2.3) may be written as:

$$\begin{aligned}
& f * \left\{h_n * \left[\frac{z}{(1-z)^2} + \frac{Mz^2}{(1-z)^2}\right]\right\} \\
&= f * \{zh_n' + M(zh_n' - h_n)\} \\
&= f * \left\{\frac{z+nz^2}{(1-z)^{n+2}} + M\left[\frac{z+nz^2}{(1-z)^{n+2}} - \frac{z}{(1-z)^{n+i}}\right]\right\} \\
&= f * \left\{\frac{z+[n+(n+1)M]z^2}{(1-z)^{n+2}}\right\} \\
&= f * \left\{\frac{z - \left[\frac{(n+1)e^{i\alpha}+[(A+nB)\cos\alpha+(n+1)iB\sin\alpha]\xi}{\cos\alpha(A-B)\xi}\right.}{(1-z)^{n+2}}\right\}
\end{aligned}$$

which completes the proof.

Theorem 1. *A necessary and sufficient condition for a function f to be in $H_n^\alpha[A,B]$ is that,*

$$1 + \sum_{k=2}^{\infty}\binom{k+n-1}{n}\left\{\frac{(1-k-kB\xi)e^{i\alpha} + (A\cos\alpha + iB\sin\alpha)\xi}{\cos\alpha(A-B)\xi}\right\}a_k z^{k-1} \neq 0 \tag{2.5}$$

for all $z \in \Delta$ and $\xi \in X$.

Proof. We notice that:

$$\frac{z-Ez^2}{(1-z)^{n+2}} = z + \sum_{k=2}^{\infty}\left\{\binom{k+n-1}{n}\frac{[n+k-(k-1)E]}{n+1}\right\}z^k \tag{2.6}$$

for any given complex number E. Now writing $E = -(n+(n+1)M)$ and using (2.6), we find that the condition (2.2) can be written as:

$$z + \sum_{k=2}^{\infty}\left\{\binom{k+n-1}{n}\frac{[n+k-(k-1)E]}{n+1}\right\}a_k z^k \neq 0 \tag{2.7}$$

On substituting the value of E together with the value of M in (2.7) and simplifying we obtain (2.5).

Corollary 1. *The function f is in $R_n[A,B]$ where $-1 \leq B < A \leq 1$, if and only if*

$$1+\sum_{k=2}^{\infty}\left\{\binom{k+n-1}{n}\frac{[(k-1)\xi+A-kB]}{A-B}\right\}a_k z^{k-1} \neq 0$$

for all $z \in \Delta$ and $\xi \in X$.

Remark. Corollary 1 is theorem 1 of [1].
Using (1.11) of Lemma 1, we can obtain the following:

Lemma 4. *The function $f \in H_n^{\alpha}(a,b)$ if and only if*

$$f * \left\{\frac{z+[n+(n+1)N]z^2}{(1-z)^{n+2}}\right\} \neq 0 \tag{2.8}$$

for all $z \in \Delta_0$, $\xi \in X$, where $N = \frac{(1+a+b\xi)\cos\alpha + i\sin\alpha}{(1-a-b\xi)\cos\alpha}$.

Theorem 2. *A necessary and sufficient condition for the function f to be in $H_n^{\alpha}(a,b)$ is that,*

$$1+\sum_{k=2}^{\infty}\left\{\binom{k+n-1}{n}\frac{[(k-a-b\xi)\cos\alpha+(k-1)i\sin\alpha]}{(1-a-b\xi)\cos\alpha}\right\}a^k z^{k-1} \neq 0 \tag{2.9}$$

for all $z \in \Delta$ and $\xi \in X$.

Proof. Letting $E = -(n+(n+1)N)$ in (2.6) and proceeding similarly as in Theorem 1, it is now easy to demonstrate that (2.8) is equivalent to (2.9) and this completes the proof.

As an application of theorem 1, we next determine a coefficient criterion for a function of the form (1.1) to be in $H_n^{\alpha}[A,B]$ (or $H_n^{\alpha}(a,b)$).

Theorem 3. *The function f of the form (1.1) is in $H_n^{\alpha}[A,B]$ if its coefficients satisfy the condition*

$$\sum_{k=2}^{\infty}\binom{K+n-1}{n}\{(k-1)+|A\cos\alpha+iB\sin\alpha-ke^{i\alpha}B|\}|a_k| \leq |A-B|\cos\alpha \tag{2.10}$$

Proof. Since

$$\left|1+\sum_{k=2}^{\infty}\binom{k+n-1}{n}\left\{\frac{(1-k-kB\xi)e^{i\alpha}+(A\cos\alpha+iB\sin\alpha)\xi}{\cos\alpha(A-B)\xi}\right\}a_k z^{k-1}\right|$$
$$\geq 1-\sum_{k=2}^{\infty}\binom{k+n-1}{n}\frac{\{(k-1)+|A\cos\alpha+iB\sin\alpha-ke^{i\alpha}B|\}|a_k|}{\cos\alpha|A-B|}$$

and the result follows from Theorem 1.

Corollary 2. *The function f is in $R_n[A,B]$ where $-1 \leq B < A \leq 1$ if its coefficients satisfy the condition.*

$$\sum_{k=2}^{\infty} \binom{k+n-1}{n} \{k-1+A-kB\}|a_k| \leq A-B$$

Remark. Corollary 2 is Theorem 3 of [1]

Corollary 3. *The function f is in $H_n^{\alpha}(a,b)$ if its coefficients satisfy the condition*

$$\sum_{k=2}^{\infty} \binom{k+n-1}{n} \Big\{(k-1)+(1-\overline{a})[(b^2+a^2-k)\cos\alpha+(1-k)i\sin\alpha]\Big\}$$
$$\leq (b^2-|1-a|^2)\cos\alpha.$$

Proof. Using (ii) of Lemma 1, the proof follows.

3. Containment Properties

In [1], Ahuja has shown that $R_{n+1}[A,B] \subset R_n[A,B]$. Now we establish similar containment results:

Theorem 4. *$H_{n+1}^{\alpha}[A,B] \subset H_n^{\alpha}[A,B]$ for all $n \in N_0$.*

Proof. If $f \in H_{n+1}^{\alpha}[A,B]$, then Theorem 1, gives:

$$1+\sum_{k=2}^{\infty} \binom{k+n}{n+1} \left\{\frac{(1-k-kB\xi)e^{i\alpha}+(A\cos\alpha+iB\sin\alpha)\xi}{\cos\alpha(A-B)\xi}\right\} a_k z^{k-1} \neq 0 \tag{3.1}$$

for all $z \in \Delta$ and $\xi \in X$. Note that (3.1) may be written as

$$\left[1+\sum_{k=2}^{\infty} \frac{k+n}{n+1} z^{k-1}\right] *$$
$$* \left[1+\sum_{k=2}^{\infty} \binom{k+n-1}{n} \left\{\frac{(1-k-kB\xi)e^{i\alpha}+(A\cos\alpha+iB\sin\alpha)\xi}{\cos\alpha(A-B)\xi}\right\} a_k z^{k-1}\right] \neq 0 \tag{3.2}$$

But

$$\left[1+\sum_{k=2}^{\infty} \frac{n+1}{k+n} z^{k-1}\right] * \left[1+\sum_{k=2}^{\infty} \frac{k+n}{n+1} z^{k-1}\right] = 1+\sum_{k=2}^{\infty} z^{k-1} = 1/(1-z)$$

for all $z \in \Delta$.
Hence form (3.2) it follows that:

$$1+\sum_{k=2}^{\infty} \binom{k+n-1}{n} \left\{\frac{(1-k-kB\xi)e^{i\alpha}+(A\cos\alpha+iB\sin\alpha)\xi}{\cos\alpha(A-B)\xi}\right\} a_k z^{k-1} \neq 0$$

for all $z \in \Delta$ and $\xi \in X$. In view of Theorem 1, it follows that $f \in H_n^\alpha[A,B]$, and this completes the proof.

Corollary 1. *$R_{n+1}[A,B] \subset R_n[A,B]$, for all $n \in N_0$.*

Corollary 2. *$H_n^\alpha[A,B] \subset S^\alpha[A,B]$, for all $n \in N_0$.*

Remark.

1. In view of corollary 2, the functions in $H_n^\alpha[A,B]$ are spirallike and hence are univalent.
2. Using the result (ii) of Lemma 1 it follows that $H_{n+1}^\alpha(a,b) \subset H_n^\alpha(a,b)$ for all $n \in N_0$.

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