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ON MATRIX SEQUENCES ON cs AND bs

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Abstract. We respectively denote the linear spaces of bounded double sequences and bounded double series by M and BS and denote the usual linear spaces of convergent single series and bounded single series by cs and bs. We determine the necessary and sufficient conditions on the sequence $\mathcal{A} = (A_p)$ of infinite matrices in the classes (cs, M), (bs, M), (cs, BS) and (bs, BS).

1. Introduction

Let s be the linear space of all single real sequences and let m, c, bv, cs and bs denote the linear spaces of bounded sequences, convergent sequences, bounded variation sequences, convergent series and bounded series, respectively. These are subspaces of s. If λ is a subset of s then we shall write λ^+ for the generalized Köthe- Toeplitz dual of λ, i.e.

$$\lambda^+ = \{a = (a_k) : ax \in cs \quad \text{for every} \quad x \in \lambda\}.$$

We define the linear double sequences spaces M and BS in the following way.

$$M = \{(x_{i,j}) \in S : \sup_{i,j\geq 0} |x_{i,j}| < \infty\},$$

$$BS = \{(x_{i,j}) \in S : \sup_{i,j\geq 0} |\sum_{k=0}^{i} x_{k,j}| < \infty\},$$

where S is the linear space of double real sequences. Obviously, if $x_{ij} = x_i$ for all j, we have $S = s$, $M = m$ and $BS = bs$.

Let $\mathcal{A}$ denote the sequence of real matrices $A_p = (a_{nk}(p))$ and let $\lambda \subset s$ and $\mu \subset S$. Then we say that the matrix sequence $\mathcal{A} = (A_p)$ defines a transformation from λ into μ, if for every sequence $x = (x_k) \in \lambda$ the double sequence $\mathcal{A}x = ((Ax)_n^p)_{n,p=0}^{\infty}$ exists and is in μ, where $(Ax)_n^p = \sum_{k=0}^{\infty} a_{nk}(p)x_k, (n, p = 0, 1, 2, \ldots)$. By (λ, μ) we denote the class

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of all such matrices. For simplicity in notation, here and after we write $\sum_k$ instead of $\sum_{k=0}^{\infty}$. If $a_{nk}(p) = a_{nk}$ for all p, then $\mathcal{A}$ is reduced to the usual summability method A and $a_{nk}(p) = 1$ $(n = k)$ for all $p; = 0(n \neq k)$ for all p then corresponds to the identity matrix I which is equivalent to the ordinary convergence. The method $\mathcal{A}$ is more general and more comprehensive than the usual summability method A.

Now throughout the paper let us write $a(n,k,p) = \sum_{j=0}^{n} a_{jk}(p) : (n,k,p = 0,1,2,\ldots)$ and we denote $y = (y_k)$ the sequence of partial sums of the series $\sum_k x_k$, i.e., $y_k = \sum_{n=0}^{k} x_n (k = 0,1,2,\ldots)$.

For $\lambda, \mu = f$ or fs, the classes (λ, μ)- matrices were characterized by Başar [1]. In this note, we consider μ as the space of double sequence space and give the necessary and sufficient conditions on the matrix sequence $\mathcal{A} = (A_p)$ in order that $\mathcal{A} \in (\lambda, \mu)$, where (λ, μ) is one of classes (cs, M), (bs, M) (cs, BS) and (bs, BS) -matrices.

2-Matrix sequences from cs, bs into M and BS.

Theorem 2.1. *$\mathcal{A} \in (cs, M)$ if and only if (a) $\sup_{n,p} \sum_k |\Delta a_{nk}(p)| < \infty$, where $\Delta a_{nk}(p) = a_{n,k}(p) - a_{n,k+1}(p)$.*

Proof. Necessity. Let $\mathcal{A} \in (cs, M)$. Since there exists $\mathcal{A}x$ for all $x \in cs$,the series $\sum_k a_{nk}(p)x_k$ converges for each $n, p = 0,1,2,\ldots$ and for all $x \in cs$. This satisfies

$$\{a_{nk}(p)\}_{k=0}^{\infty} \in cs^{+} \quad (n,p = 0,1,2,\ldots) \tag{1}$$

where $cs^{+} = bv$, [2, p69].

On the other hand, since $\mathcal{A}x \in M$ for all $x \in cs$, we have

$$\mathcal{A}e^k = \{a_{nk}(p)\}_{n,p=0}^{\infty} \in M, (k = 0,1,2,\ldots) \tag{2}$$

for $x = e^k$ (e^k is the sequences whose only non-zero term is 1 in the k-th place). The necessity of (a) is obtained by (1) and (2).

Sufficiency. Suppose that (a) holds and let $x \in cs$, $y_k \to \alpha(k \to \infty)$. Applying the Abel partial sums formula to the m-th partial sums of $\mathcal{A}x$ we have

$$(b) \qquad \sum_{k=0}^{m} a_{nk}(p)x_k = \sum_{k=0}^{m-1} \Delta a_{nk}(p)y_k + a_{nm}(p)y_m \qquad (m,n,p = 0,1,2,\ldots)$$

Since $y_m \to \alpha(m \to \infty)$, we write

$$(c) \qquad \sum_{k=0}^{m} a_{nk}(p)x_k = \sum_{k=0}^{m-1} \Delta a_{nk}(p)(y_k - \alpha) + a_{nm}(p)(y_m - \alpha) + \alpha a_{n0}(p)$$

and for $m \to \infty$ we have

$$(d) \qquad \sum_{k} a_{nk}(p)x_k = \sum_{k} \Delta a_{nk}(p)(y_k - \alpha) + \alpha a_{n0}(p).$$

On the other hand, by hypothesis, we have $\{a_{nk}(p)\}_{k=0}^{\infty} \in bv \quad (n,p = 0,1,2,\ldots)$ and $\{a_{nk}(p)\}_{n,p=0}^{\infty} \in M, (k = 0,1,2,\ldots)$. If we take supremum on n, p in the equation (d) then we have

$$\sup_{n,p\geq 0} |\sum_k a_{nk}(p)x_k| \leq \sup_{n,p\geq 0} \{\sum_k |\Delta a_{nk}(p)||y_k - \alpha| + |\alpha a_{n0}(p)|\}$$

$$\leq \|y_k - \alpha\| \sup_{n,p\geq 0} \sum_k |\Delta a_{nk}(p)| + |\alpha| \sup_{n,p\geq 0} |a_{n0}(p)| < \infty.$$

Hence $\mathcal{A}x \in M$. This completes the proof.

Theorem 2.2. *$\mathcal{A} \in (bs, M)$ if and only if*
(i) $\sup_{n,p} \sum_k |\Delta a_{nk}(p)| < \infty$,
(ii) $\lim_{k\to\infty} a_{nk}(p) = 0 (n,p = 0,1,2,\ldots)$

Proof. Necessity. Let $\mathcal{A} \in (bs, M)$. Since $cs \subset bs$, we have $(bs, M) \subset (cs, M)$ and therefore the necessity of (i) is obvious. Now to show that the necessity of (ii) we assume that (ii) is not satisfied for some n, p and we shall obtain a contradiction as in Theorem 3.1 [1]. Indeed, under this assumption we can find some $x \in bs$ such that $\mathcal{A}x$ does not belong to M. For example, if we choose $x = ((-1)^n) \in bs$ then $(\mathcal{A}x)_n^p = \sum_{k=0}^{\infty} a_{nk}(p)(-1)^k$. However, since the limit $a_{nk}(p)(-1)^k$ $(k \to \infty)$ does not even exist and is not equal to zero, the series $\sum_{k=0}^{\infty} a_{nk}(p)(-1)^k$ does not converge for each n, p. That is to say that the $\mathcal{A}$-transform of the series $\sum_{k=0}^{\infty}(-1)^k$ which belongs to bs does not even exist. But this contradicts the hypothesis. Hence (ii) is necessary.

Sufficiency. Let us suppose that (i) and (ii) hold and $x \in bs$. We always remind that $y \in m$ for $x \in bs$. Using (ii) and for $m \to \infty$, we consider (b) then

$$\sum_{k=0}^{\infty} a_{nk}(p)x_k = \sum_{k=0}^{\infty} \Delta a_{nk}(p)y_k.$$

Now if we recall (i) and take supremum on n, p in the above equality then we have

$$\sup_{n,p} |\sum_k a_{nk}(p)x_k| \leq \sup_{n,p} \sum_k |\Delta a_{nk}(p)||y_k| \leq \|y\| \sup_{n,p} \sum_k |\Delta a_{nk}(p)| < \infty.$$

This means that $\mathcal{A}x \in M$.

Thus the proof is completed.

As an immediate consequence of Theorem 2.1 and Theorem 2.2, we have

Corollary 2.1. [3] *$A \in (cs, m)$ if and only if* $\sup_n \sum_k |a_{nk} - a_{n,k+1}| < \infty$.

Corollary 2.2. [3] *$A \in (bs, m)$ if and only if*
(i) $\sup_n \sum_k |a_{nk} - a_{n,k+1}| < \infty$,
(ii) $\lim_{k\to\infty} a_{nk} = 0$ *for all* n.

Theorem 2.3. *$\mathcal{A} \in (cs, BS)$ if and only if* $\sup_{n,p} \sum_k |\Delta a(n,k,p)| < \infty$.

Proof. Let $\mathcal{A} \in (cs, BS)$ and $x \in cs$. Now we consider the following equality that has been taken by the nm-th partial sums of $(Ax)^i_j$ as $m \to \infty$.

$$\sum_{j=0}^{n} \sum_k a_{jk}(p) x_k = \sum_k a(n,k,p) x_k \quad ; (n, p = 0, 1, 2, \ldots)$$

Therefore, we have $\mathcal{A}x \in BS$ for all $x \in cs$ if and only if $Bx \in M$ for all $x \in cs$, where $B = (b_{nk}(p)) = (B_p), b_{nk}(p) = a(n,k,p)(n,k,p = 0,1,2,\ldots)$ This completes the proof.

Theorem 2.4. *$\mathcal{A} \in (bs, BS)$ if and only if*
(i) $\sup_{n,p} \sum_k |\Delta a(n,k,p)| < \infty$,
(ii) $\lim_{k\to\infty} a_{nk}(p) = 0 (n, p = 0, 1, 2, \ldots)$

Proof. This is easily obtained by the similar kind of argument of Theorem 2.3.
In the special case $\mathcal{A} = A$, by Theorem 2.3. and Theorem 2.4. we have

Corollary 2.3.[3] *$A \in (cs, bs)$ if and only if* $\sup_m \sum_k |\sum_{n=0}^m (a_{nk} - a_{n,k+1})| < \infty$.

Corollary 2.4.[3] *$A \in (bs, bs)$ if and only if*
(i) $\sup_m \sum_k |\sum_{n=0}^m (a_{nk} - a_{n,k+1})| < \infty$,
(ii) $\lim_{k\to\infty} a_{nk} = 0$ *for all* n.

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References

[1] F. Başar, "f-conservative matrix sequences," *Tamkang J. Math.*, 22(2)(1991), 205-212.
[2] P. K. Kampthan and M. Gupta, Sequence spaces and series, Marcel Dekker Inc., New York, 1981.
[3] M. Stieglitz and H. Tietz, "Matrix transformationen von folgenraümen eine ergebnisübersicht," *Math. Z.*, 154(1977), 1-16.

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