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ON THE FOURTH MOMENT OF THE MAXIMUM OF PARTIAL SUMS OF n-EXCHANGEABLE RANDOM VARIABLES WITH APPLICATIONS

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Abstract. A general formula is obtained for the fourth moment of the maximum of partial sums of n-exchangeable random variables derived from a result of Spitzer. The formula is applied in particular to obtain the fourth moment of the maximum of adjusted partial sums of normal summands. This is of direct relevance to reservoir design and the analysis of the structure of stochastic processes and time series.

I. Introduction

Let $Y_1, Y_2, \ldots, Y_n$ be n-exchangeable random varibles. Define

$$S_r = Y_1 + Y_2 + \cdots + Y_r, \quad r = 1, 2, \ldots, n$$

and $M_n = \max(0, S_1, S_2, \ldots, S_n) = \max_{1\le k\le n}(S_k^+)$, where $S_k^+ = \max(0, S_k)$.

Spitzer [9] investigated aspects of the distribution of M_n and proved that the expected value of any integrable complex valued function $f(M_n)$ satisfies:

$$E[f(M_n)] = \frac{1}{n!}\sum_{\tau} E\{f[T(\tau Y)]\}, \tag{1.1}$$

where τY is any one of the $n!$ permutations of $Y_1, Y_2, \ldots, Y_n$, $T(\tau Y) = \sum_{i=1}^{n(\tau)}(\sum_{k\in\alpha_i} Y_k)^+$ and $\tau = (\alpha_1)(\alpha_2)\ldots(\alpha_{n(\tau)})$ is a permutation represented as a product of cycles α_i including the one-cycle and with no index contained in more than one cycle.

Using (1.1), selecting $f(M_n)$ to be $\exp(i\lambda M_n)$ and considering the case when $Y_1, Y_2, \ldots, Y_n$ are independent identically distributed (for the exchangeability of $Y_1, Y_2, \ldots, Y_n$

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in this case see [4]) Spitzer derived an expression (Theorem 3.5 in [9]) relating the characteristic function of M_n to that of S_k^+ from which, in turn, he deduced the first moment of M_n. Solari and Dunnage [7] showed, explicitly, that this formula for $E(M_n)$ is also valid for the general case of exchangeable random variables.

After that, on the basis of (1.1), and following Spitzer's approach, Ains and Gharib [2] and Gharib [6] derived, respetively, formulas for $E(M_n^2)$ and $E(M_n^3)$ for the general case of exchangeable random variables.

In the present work this process is carried out one step forward by obtaining the fourth moment $E(M_n^4)$ in the general case of exchangeable summands $Y_1, Y_2, \ldots, Y_n$. We show that, for $n \geq 4$,

$$\begin{aligned} E(M_n^4) = &\sum_{k=1}^{n} \frac{E(S_k^{+4})}{k} + \sum_{r=1}^{n-1}\sum_{k=1}^{r} \frac{E(4S_k^{+3}(S_{r+1}-S_k)^+ + 3S_k^{+2}(S_{r+1}-S_k)^{+2}]}{k(r+1-k)} \\ &+ 6\sum_{r=1}^{n-2}\sum_{k=1}^{r}\sum_{t=1}^{k} \frac{E[S_t^+(S_{k+1}-S_t)^+(S_{r+2}-S_{k+1})^{+2}]}{t(k+1-t)(r+1-k)} \\ &+ \sum_{r=1}^{n-3}\sum_{k=1}^{r}\sum_{t=1}^{k}\sum_{l=1}^{t} \frac{E[S_l^+(S_{t+1}-S_l)^+(S_{k+2}-S_{t+1})^+(S_{r+3}-S_{k+2})^+]}{l(t+1-l)(k+1-t)(r+1-k)}, \end{aligned} \tag{1.2}$$

Due to the fact that the problem of obtaining the exact formula of the probability distribution of the maximum M_n is of unmanageable complexity (see [3]), formula (1.2) besides that of $E(M_n^2)$ and $E(M_n^3)$ enables us to study the characteristics (such as skewness and kurtosis) of the distribution of M_n for the cases of practical importance.

As a special case, let $Y_1, Y_2, \ldots, Y_n$ be n-independent standard normal variates. We show in this case that

$$\begin{aligned} E(M_n^4) = &\frac{3}{8}n(3n+1) + 4\sqrt{2/\pi}\sum_{r=1}^{n-1}\sum_{k=1}^{n-r}\sqrt{r/k} + \frac{3}{2\pi}\sum_{r=2}^{n-1}\sum_{k=1}^{r-1}\frac{(n-r)}{\sqrt{k(r-k)}} \\ &+ (2\pi)^{-2}\sum_{r=1}^{n-3}\sum_{k=1}^{r}\sum_{t=1}^{k}\sum_{l=1}^{t}[l(t+1-l)(k+1-t)(r+1-k)]^{-1/2}. \end{aligned} \tag{1.3}$$

In this particular case a recurrence relation (equation (3.10)) is obtained for the moments $E(M_n^j)$, $j = 1, 2\ldots$ which makes possible their numerical calculations.

A second application is also given to the exchangeable random variables $Y_1, Y_2, \ldots, Y_n$ defined by

$$Y_i = X_i - \overline{X}, \quad i = 1, 2, \ldots, n$$

where $X_1, X_2, \ldots, X_n$ are n-independent standard normal variates and $n\overline{X} = \sum_{i=1}^{n} X_i$.

2. Proof of (1.2)

Putting $f(x) = x^4$ in (1.1) we obtain

$$E(M_n^4) = \frac{1}{n!} \sum_{\tau} E\{[T(\tau Y)]^4\} \tag{2.1}$$

Now, if a permutation τ consists of K_ν cycles of length ν, $\nu = 1, 2, \ldots, n$ with $\sum_{\nu=1}^{n} \nu K_\nu = n$, then

$$E\{[T(\tau Y)]^4\} = T_1 + 4T_2 + 3T_3 + 6T_4 + T_5, \tag{2.2}$$

where

$$\begin{aligned} T_1 = \sum_{\nu=1}^{n} \Big\{ & k_\nu E(S_\nu^{+4}) + k_\nu(k_\nu - 1)[4E[S_\nu^{+3}(S_{2\nu} - S_\nu)^+] + 3E[S_\nu^{+2}(S_{2\nu} - S_\nu)^{+2}] \\ & + 6k_\nu(k_\nu - 1)(k_\nu - 2)E[S_\nu^{+2}(S_{2\nu} - S_\nu)^+(S_{3\nu} - S_{2\nu})^+] \\ & + k_\nu(k_\nu - 1)(k_\nu - 2)(k_\nu - 3)E[S_\nu^+(S_{2\nu} - S_\nu)^+(S_{3\nu} - S_{2\nu})^+(S_{4\nu} - S_{3\nu})^+]\Big\}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} T_2 = \sum\sum_{\mu \neq \nu}^{n} \Big\{ & k_\nu k_\mu E[S_\nu^{+3}(S_{\nu+\mu} - S_\nu)^+] + 3k_\mu k_\nu(k_\nu - 1)E[S_\nu^{+2}(S_{2\nu} - S_\nu)^+(S_{2\nu+\mu} - S_{2\nu})^+] \\ & + k_\mu k_\nu(k_\nu - 1)(k_\nu - 2)E[S_\nu^+(S_{2\nu} - S_\nu)^+(S_{3\nu} - S_{2\nu})^+(S_{3\nu+\mu} - S_{3\nu})^+]\Big\}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} T_3 = \sum\sum_{\mu \neq \nu}^{n} \Big\{ & k_\mu k_\nu E[S_\nu^{+2}(S_{\mu+\nu} - S_\nu)^{+2}] + k_\mu k_\nu(k_\nu - 1)E[S_\nu^+(S_{2\nu} - S_\nu)^+(S_{\mu+2\nu} - S_{2\nu})^{+2}] \\ & + k_\nu k_\mu(k_\mu - 1)E[S_\nu^+(S_{\mu+\nu} - S_\nu)^+(S_{2\mu+\nu} - S_{\mu+\nu})^+] \\ & + k_\nu k_\mu(k_\nu - 1)(k_\mu - 1)E[S_\nu^+(S_{2\nu} - S_\nu)^+(S_{2\nu+\mu} - S_{2\nu})^+(S_{2\nu+2\mu} - S_{2\nu+\mu})^+]\Big\}, \end{aligned} \tag{2.5}$$

$$\begin{aligned} T_4 = \sum\sum\sum_{\nu \neq \mu \neq \lambda}^{n} \Big\{ & k_\nu k_\mu k_\lambda E[S_\nu^{+2}(S_{\nu+\mu} - S_\nu)^+(S_{\nu+\mu+\lambda} - S_{\nu+\mu})^+] \\ & + k_\nu k_\mu k_\lambda(k_\nu - 1)E[S_\nu^+(S_{2\nu} - S_\nu)^+(S_{2\nu+\mu} - S_{2\nu})^+(S_{2\nu+\mu+\lambda} - S_{2\nu+\mu})^+\Big\} \end{aligned} \tag{2.6}$$

and

$$T_5 = \sum\sum\sum\sum_{\nu \neq \mu \neq \lambda \neq \rho}^{n} k_\nu k_\mu k_\lambda k_\rho E[S_\nu^+(S_{\mu+\nu} - S_\nu)^+(S_{\lambda+\mu+\nu} - S_{\mu+\nu})^+(S_{p+\lambda+\mu+\nu} - S_{\lambda+\mu+\nu})^+] \tag{2.7}$$

Now, the number of permutations on n objects which when decomposed into disjoint cycles exhibit the above structure is exactly

$$n! \prod_{\nu=1}^{n} \nu^{-k_\nu}(K_\nu!)^{-1} \tag{2.8}$$

Therefore, from (2.1), (2.2) and (2.8) we have

$$E(M_n^4) = \Sigma^* \prod_{\nu=1}^{n} \nu^{-k_\nu}(K_\nu!)^{-1}[T_1 + 4T_2 + 3T_3 + 6T_4 + T_5], \tag{2.9}$$

where the summation Σ^* -extends over all n-tuples $(K_1, K_2, \ldots, K_n)$ of non-negative integers with the property $\sum_{i=1}^{n} iK_i = n$.

Finally , from (2.3)-(2.7) and (2.9) on reordering the summations and making use of the following set of identities, we get (1.2).

$$\Sigma^* k_\nu(k_\nu - 1)\cdots(k_\nu - r)\prod_{\nu=1}^{n}\nu_\nu^{-k_\nu}(k_\nu!)^{-1} = \begin{cases}\nu^{-(r+1)} & ,(r+1)\nu \le n \\ 0 & ,(r+1)\nu > n,\end{cases} \quad r=0,1,2,3 \tag{2.10}$$

$$\Sigma^* k_\nu(k_\nu - 1)\cdots(k_\nu - r)k_\mu(k_\mu - 1)\cdots(k_\mu - s)\prod_{\nu=1}^{n}\nu^{-k_\nu}(k_\nu!)^{-1}$$
$$= \begin{cases}\nu^{-(r+1)}\mu^{-(s+1)} & ,\mu \neq \nu,(r+1)\nu + (s+1)\mu \le n \\ 0 & ,\mu \neq \nu,\ (r+1)\nu + (s+1)\mu > n,\end{cases} \quad r = 0,1,2; s = 0,1; r+s \le 2 \tag{2.11}$$

$$\Sigma^* k_\lambda k_\mu k_\nu(k_\nu - 1)\cdots(k_\nu - r)\prod_{\nu=1}^{n}\nu^{-k_\nu}(k_\nu!)^{-1}$$
$$= \begin{cases}(\lambda\mu\nu^{r+1})^{-1} & ,\lambda \neq \mu \neq \nu,\ (r+1)\nu + \lambda + \mu \le n \\ 0 & ,\lambda \neq \mu \neq \nu,\ (r+1)\nu + \lambda + \mu > n,\end{cases} \quad r = 0,1. \tag{2.12}$$

$$\Sigma^* k_p k_\lambda k_\mu k_\nu \prod_{\nu=1}^{n}\nu^{-k_\nu}(k_\nu!)^{-1} = \begin{cases}(p\lambda\mu\nu)^{-1} & ,p \neq \lambda \neq \mu \neq \nu,\ p+\lambda+\mu+\nu \le n \\ 0 & ,p \neq \lambda \neq \mu \neq \nu,\ p+\lambda+\mu+\nu > n\end{cases} \tag{2.13}$$

Identities (2.10)-(2.13) can be proved using mathematical induction (see [2], [6] and [8]).

3. First Application to the Normal Case

We now apply formula (1.2) to the case where $Y_1, Y_2, \ldots, Y_n$ are n-independent standard normal variates. In this case, it is easy to see that:

(i) $S_k \sim N(0, K)$. Hence,

$$E(S_K^{+4}) = \frac{3}{2}K^2 \tag{3.1}$$

(ii) S_k, $S_{r+1}(k \le r)$ have a bivariate normal distribution with zero means, variances k and $(r+1)$, repectively, and correlation coefficient $(k/(r+1))^{1/2}$.
It follows that

$$E[S_k^{+2}(S_{r+1} - S_k)^{+2}] = \frac{1}{4}k(r+1-k), \tag{3.2}$$

and

$$E[S_k^{+3}(S_{r+1} - S_k)^{+}] = \frac{1}{\pi}[k^3(r+1-k)]^{1/2} \tag{3.3}$$

(iii) S_t, S_{k+1}, $S_{r+2}(t \le k \le r)$ have a trivariate normal distribution with zero means and dispersion matrix:

$$\begin{bmatrix} t & t & t \\ t & k+1 & k+1 \\ t & k+1 & r+2 \end{bmatrix}$$

Hence, it is easy to see that:

$$E[S_t^+(S_{k+1}-S_t)^+(S_{r+2}-S_{k+1})^{+2}] = \frac{1}{4\pi}(r+1-k)[t(k+1-t)]^{-1/2}, \qquad (3.4)$$

and

(iv) $S_l, S_{t+1}, S_{k+2}, S_{r+3}$ $(l \le t \le k \le r)$ have a four-variate normal distribution with zero means and dispersion matrix

$$\begin{bmatrix} l & l & l & l \\ l & t+1 & t+1 & t+1 \\ l & t+1 & k+2 & k+2 \\ l & t+1 & k+2 & r+3 \end{bmatrix}$$

Consequently, on integrating, we get:

$$\begin{aligned} &E[S_l^+(S_{t+1}-S_l)^+(S_{k+2}-S_{t+1})^+(S_{r+3}-S_{k+2})^+] \\ &=((2\pi)^{-2}[l(t+1-l)(k+1-t)(r+1-k)]^{1/2} \end{aligned} \qquad (3.5)$$

Finally, on substituting from (3.1)-(3.5) into (1.2) we get, after some simple algebra, that

$$\begin{aligned} E[m_n^4] =& \frac{3}{8}n(3n+1) + 4\sqrt{2/\pi}\sum_{r=1}^{n-1}\sum_{k=1}^{n-r}(r/k)^{1/2} + \frac{3}{2\pi}\sum_{r=2}^{n-1}\sum_{k=1}^{r-1}\frac{(n-r)}{\sqrt{k(r-k)}} \\ &+ (2\pi)^{-2}\sum_{r=1}^{n-3}\sum_{k=1}^{r}\sum_{t=1}^{k}\sum_{l=1}^{t}[l(t+1-l)(k+1-t)(r+1-k)]^{-1/2} \end{aligned} \qquad (3.6)$$

A Recurrence Relation for the moments of M_n:
If we define

$$M_n^* = \max_{1 \le k \le n}(S_k),$$

then it is easy to see that

$$M_n^* = M_{n-1} + X_n$$

OR, for a sample of size $(n+1)$,

$$M_{n+1}^* = M_n + X_{n+1} \qquad (3.7)$$

Consequently, (since X_{n+1} is a standard normal variate, independent of M_n),

$$\begin{aligned} E(M^*_{n+1}) &= E(M_n), \\ E(M^{*2}_{n+1}) &= E(M^2_n) + 1, \\ E(M^{*3}_{n+1}) &= E(M^3_n) + 3E(M_n), \\ E(M^{*4}_{n+1}) &= E(M^4_n) + 6E(M^2_n) + 3, \ldots \text{ etc.} \end{aligned} \tag{3.8}$$

Comparing our (3.8) with (2.7) of [1], we may conclude easily that the moments of M_n in the standard normal case are, in fact, the functions $M_j(n+1)$ defined by (2.5) in [1]. Viz.

$$E(M^j_n) = M_j(n+1), \qquad j = 1, 2, \ldots \tag{3.9}$$

Therefore $E(M^j_n)$ must satisfy the recurrence relation of $M_j(n+1)$ given by (6.3) in [1]. Hence

$$E(M^j_n) = \sum_{r=1}^{n-1} (2\pi r)^{-1/2} E(M^{j-1}_{n-r}) + n(j-1)E(M^{j-2}_n) - \frac{1}{2}(j-1) \sum_{r=1}^{n-1} E(M^{j-2}_r), \ j \geq 2 \tag{3.10}$$

If we put $j = 4$ in (3.10) we can get (3.6) on using the results of [2] and [6] concerning, respectively, the second and the third moments of M_n in this case. The recurrence relation (3.10) gives the ability for the numerical computations of the moments of M_n.

4. Second Application to the Normal Case

Let $Y_i = X_i - \overline{X}, \quad i = 1, 2, \ldots, n,$
where $X_1, X_2, \ldots, X_n$ are n-independent standard normal variates. For the exchangeability of the Y_i's in this case see [4].

Since $S_n = \sum_{i=1}^n Y_i = 0$, relation (1.2) remains valid but with n replaced by $(n-1)$.

Because $X_1, X_2, \ldots, X_n$ are independent standard normal variates, it is not difficult to verify that:

(i) S_k is normal with mean zero and variance $k(n-k)/n$, and thus

$$E(S^{+4}_k) = \frac{3}{2}[k(n-k)/n]^2 \tag{4.1}$$

(ii) S_k, S_{r+1} $(k \leq r)$ have a bivariate normal distribution with zero means, variances $k(n-k)/n$, and $(r+1)(n-r-1)/n$, respectively, and correlation coefficient

$$[k(n-r-1)/(r+1)(n-k)]^{1/2}.$$

Hence, routine integration, gives

$$\begin{aligned} \frac{E[S^{+3}_k(S_{r+1} - S_k)^+]}{k(r+1-k)} =& \frac{3}{2\pi} \frac{k(n-k)}{n^2} \Big[[\frac{n(n-r-1)}{k(r+1-k)}]^{1/2} - \tan^{-1} [\frac{n(n-r-1)}{k(r+1-k)}]^{1/2} \Big] \\ & - \frac{1}{2\pi} \frac{(n-r-1)}{(n+k-r-1)} [\frac{k(n-r-1)}{n(r+1-k)}]^{1/2}, \end{aligned} \tag{4.2}$$

and,

$$\frac{E[S_k^{+2}(S_{r+1}-S_k)^{+2}]}{k(r+1-k)} = \frac{1}{2\pi n^2}[n(n-r-1)+3k(r+1-k)]\tan^{-1}\left[\frac{n(n-r-1)}{k(r+1-k)}\right]^{1/2} - \frac{3}{2\pi}[k(r+1-k)(n-r-1)/n^3]^{1/2} \tag{4.3}$$

(iii) S_t, S_{k+1}, S_{r+2}, $(t \le k \le r)$ have a trivariate normal distribution with zero means and dispersion matrix

$$\frac{1}{n}\begin{bmatrix} t(n-t) & t(n-k-1) & t(n-r-2) \\ t(n-k-1) & (k+1)(n-k-1) & (k+1)(n-r-2) \\ t(n-r-2) & (k+1)(n-r-2) & (r+2)(n-r-2) \end{bmatrix}$$

Consequently, on integrating, we get

$$\begin{aligned} &\frac{E[S_t^+(S_{k+1}-S_t)^+(S_{r+2}-S_{k+1})^{+2}]}{t(k+1-t)(r+1-k)} \\ &= \frac{1}{4\pi}[n^3(n-r-2)^2]^{-1/2}\{[n(n-r-2)+2(k+1)(r+1-k)][t(k+1-t)(n-k-1)]^{-1/2} \\ &- [(r+1-k)(2n-k-1)/(n-t)(n+t-k-1)][t(k+1-t)/(n-k-1)]^{1/2} \\ &- [(2n-3t)/(n-t)][(r+1-k)(n+k-r-t-1)/t]^{1/2} \\ &- [(2n-3(k+1-t))/(n+t-k-1)][(r+1-k)(n+t-r-2)/(k+1-t)]^{1/2} \\ &- [[n-3(r+1-k)]/(\sqrt{n})]\{\tan^{-1}[n(n+t-r-2)/(k+1-t)(r+1-k)]^{1/2} \\ &+ \tan^{-1}[n(n+k-r-t-1)/t(r+1-k)]^{1/2} \\ &- (\pi - \tan^{-1}[n(n-k-1)/t(k+1-t)]^{1/2})\}\} \end{aligned} \tag{4.4}$$

and

(iv) $S_l, S_{t+1}, S_{k+2}, S_{r+3}$, $(l \le t \le k \le r)$ have a four-variate normal distribution with zero means and dispersion matrix

$$\frac{1}{n}\begin{bmatrix} l(n-l) & l(n-t-1) & l(n-k-2) & l(n-r-3) \\ l(n-t-1) & (t+1)(n-t-1) & (t+1)(n-k-2) & (t+1)(n-r-3) \\ l(n-k-2) & (t+1)(n-k-2) & (k+2)(n-k-2) & (k+2)(n-r-3) \\ l(n-r-3) & (t+1)(n-r-3) & (k+2)(n-r-3) & (r+3)(n-r-3) \end{bmatrix}$$

Hence, it is easy to see that

$$\begin{aligned} &E[S_l^+(S_{t+1}-S_l)^+(S_{k+2}-S_{t+1})^+(S_{r+3}-S_{k+2})^+] \\ &= C\frac{(r+1-k)(n-r-3)}{(n-k-2)}\left[I_1(n,l,t,k,r) - \left[\frac{2\pi(r+1-k)}{(n-k-2)(n-r-3)}\right]^{1/2} I_2(n,l,t,k,r)\right], \end{aligned} \tag{4.5}$$

where

$$C = \frac{n^{1/2}}{4\pi^2}[l(t+1-l)(k+1-t)(r+1-k)(n-r-3)]^{-1/2}, \tag{4.6}$$

$$I_1(n,l,t,k,r) = \int_{x=0}^{\infty}\int_{y=x}^{\infty}\int_{z=y}^{\infty} x(y-x)(z-y)\exp\Big[-\frac{1}{2}\frac{(t+1)}{l(t+1-l)}x^2 - \frac{2}{(t+1-l)}xy + \frac{(k+2-l)}{(k+1-t)(t+1-l)}y^2 - \frac{2}{(k+1-t)}yz + \frac{(n+k-r-t-2)}{(k+1-t)(n-r-3)}z^2\Big]dzdydx, \quad (4.7)$$

and

$$I_2(n,l,t,k,r) = \int_{x=0}^{\infty}\int_{y=x}^{\infty}\int_{z=y}^{\infty} xz(y-x)(z-y)\phi\Big(-z\sqrt{\frac{r+1-k}{(n-k-2)(n-r-3)}}\Big) \exp\Big[-\frac{1}{2}\Big[\frac{(t+1)}{l(t+1-l)}x^2 - \frac{2}{(t+1-l)}xy + \frac{(k+2-l)}{(k+1-t)(t+1-l)}y^2 - \frac{2}{(k+1-t)}yz + \frac{(n-t-1)}{(k+1-t)(n-k-2)}z^2\Big]\Big]dzdydx, \quad (4.8)$$

and $\phi(\cdot)$ is the standard normal distribution function.

Now, the integral $I_2(n,l,t,k,r)$ has no exact value, however, using a well known result (see [5]) concerning the function $\phi(\cdot)$ we get that

$$I_2(n,l,t,k,r) \sim \Big[\frac{(n-k-2)(n-r-3)}{(r+1-k)}\Big]^{1/2} I_1(n,l,t,k,r) \quad (4.9)$$

Hence, from (4.5) and (4.9), we get

$$E[S_l^+(S_{t+1}-S_l)^+(S_{k+2}-S_{t+1})^+(S_{r+3}-S_{k+2})^+] \simeq C\frac{(r+1-k)(n-r-3)}{(n-k-2)}(1-\sqrt{2\pi})I_1(n,l,t,k,r) \quad (4.10)$$

Now, using routine integration we get that

$$I_1(n,l,t,k,r) = \Big[\frac{l(t+1-l)(k+1-t)(n-r-3)}{(n+k-r-1)^2}\Big]\Big\{(n+k-r-1)-[(n+k-r-1)-2(k+1-t)] \times \Big[\frac{l(t+1-l)}{(n-r-3)(n+t-r-2)}\Big]^{1/2}\tan^{-1}\Big[\frac{(n-r-3)(n+t-r-2)}{l(t+1-l)}\Big]^{1/2} - [(n+k-r-1)-2(t+1-l)]\Big[\frac{l(k+1-t}{(n-r-3)(n+k+l-r-t-2)}\Big]^{1/2} \times \tan^{-1}\Big[\frac{(n-r-3)(n+k+l-r-t-2)}{l(k+1-t)}\Big]^{1/2} - [(n+k-r-1)-2l]\Big[\frac{(t+1-l)(k+1-t)}{(n-r-3)(n+k-r-l-1)}\Big]^{1/2} \times \tan^{-1}\Big[\frac{(n-r-3)(n+k-r-l-1)}{(t+1-l)(k+1-t)}\Big]^{1/2}\Big\} \quad (4.11)$$

Therefore, from (4.6), (4.10) and (4.11) we get

$$\frac{E[S_l^+(S_{t+1}-S_l)^+(S_{k+2}-S_{t+1})^+(S_{r+3}-S_{k+2})^+]}{l(t+1-l)(k+1-t)(r+1-k)} \simeq [(1-\sqrt{2\pi}/4\pi^2]C_n(l,t,k,r), \tag{4.12}$$

where,

$$\begin{aligned}
&C_n(l,t,k,r)\\
=&\Big[\frac{(n-r-3)}{(n-k-2)(n+k-r-1)^2}\Big]\Big[\frac{n(n-r-3)}{l(t+1-l)(k+1-t)(r+1-k)}\Big]^{1/2}\\
&\Big\{(n+k-r-1)-[(n+k-r-1)-2(k+1-t)]\Big[\frac{l(t+1-l)}{(n-r-3)(n+t-r-2)}\Big]^{1/2}\\
&\tan^{-1}\Big[\frac{(n-r-3)(n+t-r-2)}{l(t+1-l)}\Big]^{1/2}-[(n+k-r-1)-2(t+1-l)]\\
&\Big[\frac{l(k+1-t)}{(n-r-3)(n+k+l-r-t-2)}\Big]^{1/2}\tan^{-1}\Big[\frac{(n-r-3)(n+k+l-r-t-2)}{l(k+1-t)}\Big]^{1/2}\\
&-[(n+k-r-1)-2l]\Big[\frac{(t+1-l)(k+1-t)}{(n-r-3)(n+k-r-l-1)}\Big]^{1/2}\\
&\tan^{-1}\Big[\frac{(n-r-3)(n+k-r-l-1)}{(t+1-l)(k+1-t)}\Big]^{1/2}\Big\}
\end{aligned} \tag{4.13}$$

Finally, from (1.2) (with n replaced by $(n-1)$), (4.1)-(4.4), and (4.12) we get

$$\begin{aligned}
E(M_n^4) \simeq&\frac{3}{8}(n-1)^2+\frac{1}{2\pi n^2}\sum_{r=2}^{n-1}\sum_{k=1}^{r-1}A_n(k,r)+\frac{3}{2\pi}n^{-3/2}\sum_{r=2}^{n-2}\sum_{k=1}^{r-1}\sum_{t=1}^{k}B_n(t,k,r)\\
&+[(1-\sqrt{2\pi})/4\pi^2]\sum_{r=1}^{n-4}\sum_{k=1}^{r}\sum_{t=1}^{k}\sum_{l=1}^{t}C_n(l,t,k,r),
\end{aligned} \tag{4.14}$$

where,

$$\begin{aligned}
A_n(k,r)=&[(n+k-r)(8n-9r+6k)-4k(r-k)][nk(n-r)/(r-k)(n+k-r)^2]^{1/2}\\
&-3[nr+k(4n-3r-k)]\tan^{-1}[n(n-r)/k(r-k)]^{1/2},
\end{aligned}$$

$$\begin{aligned}
&(n-r-1)B_n(t,k,r)\\
=&[n(n-r-1)+2(k+1)(r-k)][t(k+1-t)(n-k-1)]^{-1/2}\\
&-\Big[\frac{(r-k)(2n-k-1)}{(n-t)(n+t-k-1)}\Big]\Big[\frac{t(k+1-t)}{(n-k-1)}\Big]^{1/2}-\Big[\frac{(2n-3t)}{(n-t)}\Big]\Big[\frac{(r-k)(n+k-r-t)}{t}\Big]^{1/2}\\
&-\Big[\frac{(2n-3(k+1-t)}{(n+t-k-1)}\Big]\Big[\frac{(r-k)(n+t-r-1)}{(k+1-t)}\Big]^{1/2}\\
&-\Big[\frac{(n-3(r-k))}{\sqrt{n}}\Big]\Big\{\tan^{-1}\Big[\frac{n(n+k-r-1)}{(r-k)(k+1-t)}\Big]^{1/2}\\
&+\tan^{-1}\Big[\frac{n(n+k-r-t)}{t(r-k)}\Big]^{1/2}-(\pi-\tan^{-1}\Big[\frac{n(n-k-1)}{t(k+1-t)}\Big]^{1/2})\Big\},
\end{aligned}$$

and $C_n(l,t,k,r)$ is given by (4.13).

The following table gives for $n = 5(1)15$ the moments about the mean μ_2, μ_3, μ_4 of M_n and the moment ratios $\gamma_1 = \mu_3/\mu_2^{3/2}$, $\gamma_2 = (\mu_4/\mu_2^2) - 3$, by using (4.14) and the formulate of $E(M_n)$, $E(M_n^2)$ and $E(M_n^3)$ given, respectively, in [7], [2] and [6].

n	μ_2	μ_3	μ_4	γ_1	γ_2
5	0.5270	−1.5098	5.9107	−3.9468	18.2854
6	0.6377	−2.4470	10.5024	−4.8056	22.8294
7	0.7473	−3.5385	16.6132	−5.4778	26.7515
8	0.8562	−4.7702	24.2961	−6.0210	30.1419
9	0.9647	−6.1314	33.5894	−6.4707	33.0904
10	1.0730	−7.6133	44.5223	−6.8501	35.6732
11	1.1810	−9.2090	57.1181	−7.1754	37.9529
12	1.2889	−10.9125	71.3960	−7.4578	39.9795
13	1.3966	−12.7185	87.3717	−7.7058	41.7930
14	1.5043	−14.6227	105.0590	−7.9254	43.4260
15	1.6119	−16.6211	124.4695	−8.1217	44.9046

It will be seen that as n increases, the distribution of M_n becomes increasingly asymmetrical and leptokurtic.

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