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LOWER BOUND TO A PROBLEM OF MOCANU ON DIFFERENTIAL SUBORDINATION

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Abstract. Let $\mathcal{S}^*$ denote the family of starlike mappings in the unit disc Δ. Let $\mathcal{R}(\alpha,\beta)$ denote the family of normalized analytic functions in Δ satisfying the condition $\mathrm{Re}(f'(z)+\alpha zf''(z))>\beta$, $z\in\Delta$ for some $\alpha>0$. In this note, among other things, we give a lower bound to the problem of Mocanu aimed at determining $\inf\{\alpha: R(\alpha,0)\subset\mathcal{S}^*\}$.

1. Introduction and Preliminaries

We denote the unit disc in the complex plane, that is $\{z\in\mathbb{C}:|z|<1\}$, by Δ. The family $\mathcal{A}$ is the family of all functions f which are analytic in the unit disc Δ and which satisfiy $f(0)=f'(0)-1=0$. Let us denote by $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{S}$ the well-known sub families of $\mathcal{A}$ consisting of starlike, close-to-convex and univalent functions respectively. A sufficient condition for $f\in\mathcal{A}$ to be in $\mathcal{C}$ is that $\mathrm{Re}\, f'(z)>0$ in Δ. In 1986 Mocanu [3] proved the following:

$$\mathcal{R}(\alpha,0)\subset\mathcal{S}^* \text{ at least when } \alpha\geq 1/2 \tag{1.1}$$

where

$$\mathcal{R}(\alpha,\beta)=\{f\in\mathcal{A}:\mathrm{Re}(f'(z)+\alpha zf''(z))>\beta,\ z\in\Delta\}.$$

Further, in the same article Mocanu raised the problem of determining

$$\inf\{\alpha:\mathcal{R}(\alpha,0)\subset\mathcal{S}^*\}.$$

However, in [4] the author improved the result (1.1) in the following way:

$$\mathcal{R}(\alpha,\beta(0,\alpha,\alpha'))\subset\mathcal{S}^* \quad \text{at least when } \alpha\geq\alpha'\approx 0.426 \tag{1.2}$$

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where

$$\beta(\delta,\alpha,\alpha') = \frac{\delta - (1-\alpha'/\alpha)(2\rho(\alpha)-1)}{1-(1-\alpha'/\alpha)(2\rho(\alpha)-1)} \tag{1.3}$$

and

$$\rho(\alpha) = \int_0^1 \frac{dt}{1+t^\alpha}. \tag{1.4}$$

The purpose of this note is to determine a lower bound α' to the problem of Mocanu such that

$$\mathcal{R}(\alpha, \beta(0,\alpha,\alpha')) \subset \mathcal{S}^* \quad \text{for } \alpha \geq \alpha' \approx 0.4036.$$

Connected to this a ralated but general result is also derived.

For the proof of our result we need the following lemmas:

Lemma 1.5.([5]) *Suppose that* $\alpha \geq \alpha' > 0$, $\delta < 1$ *and* $\beta(\delta,\alpha,\alpha')$ *is defined by* (1.3). *Then*

$$\mathcal{R}(\alpha, \beta(\delta,\alpha,\alpha')) \subset \mathcal{R}(\alpha',\delta).$$

Lemma 1.6. ([6, Lemma 1]) *Let* β_0 *be the solution of*

$$\beta\pi = \frac{3\pi}{2} - \arctan\eta,$$

for a suitable fixed $\eta > 0$ *so that* $\lambda(z) : \Delta \longrightarrow \mathbb{C}$ *satisfies*

$$|\mathrm{Im}\,\lambda(z)| \leq \frac{1}{\eta}\left(\mathrm{Re}\,\lambda(z) - \frac{\eta}{\beta}\right), \quad z \in \Delta$$

and let

$$\alpha = \alpha(\beta,\eta) = \beta + \frac{2}{\pi}\arctan\eta, \quad 0 < \beta \leq \beta_0.$$

If p *is analytic in* Δ *with* $p(0) = 1$ *and satisfies*

$$|\arg(p(z) + \lambda(z)zp'(z))| < \frac{\pi}{2}\alpha, \quad z \in \Delta$$

then

$$|\arg p(z)| < \frac{\pi}{2}\beta, \quad z \in \Delta.$$

Lemma 1.7. ([1, 2, 3]) *Let* Ω *be a set in the complex plane* $\mathbb{C}$ *and suppose that* $\psi : \mathbb{C}^2 \times \Delta \longrightarrow \mathbb{C}$ *satisfies the condition*

$$\psi(ix, y; z) \notin \Omega, \ \textit{for } z \in \Delta \textit{ and real } x, y \textit{ with } y \leq -(1+x^2)/2. \tag{1.8}$$

If p *is analytic in* Δ, $p(0) = 1$, *and if* $\psi(p(z), zp'(z); z) \in \Omega$ *then* $\mathrm{Re}\,p(z) > 0$ *for* $z \in \Delta$.

All of the inequalities in this article involving functions of z hold uniformly in the unit disc Δ. Therefore, the condition '$z \in \Delta$' will be omitted in the remainder of the article.

2. Lower Bound to the Problem of Mocanu

Lemma 1.6 has found a number of interesting applications in [5]. Apart from these, a very important and useful special case yields the following theorem.

Theorem 2.1. *If p is analytic in Δ, with $p(0) = 1$, and satisfies*

$$\operatorname{Re}(p(z) + \alpha z p'(z)) > -\frac{(1-\alpha'/\alpha)(2\rho(\alpha)-1)}{1-(1-\alpha'/\alpha)(2\rho(\alpha)-1)} \tag{2.2}$$

for some $\alpha \geq \alpha' \equiv 3\cot((1+\varepsilon)\pi/3)/(2(1+\varepsilon))$, $-1 < \varepsilon \leq 1/2$, then

$$|\arg p(z)| < \frac{(1+\varepsilon)\pi}{3},$$

where $\rho(\alpha)$ is defined by (1.4).

Proof. In view of Lemma 1.5, with $p(z) = f'(z)$, and (2.2) it suffices to show that

$$\operatorname{Re}(p(z) + \alpha' z p'(z)) > 0 \text{ implies } |\arg p(z)| < \frac{(1+\varepsilon)\pi}{3}. \tag{2.3}$$

From the condition on α' and ε, it is easy to see that

$$1 = \frac{2(1+\varepsilon)}{3} + \frac{2}{\pi}\arctan\left(\frac{2\alpha'(1+\varepsilon)}{3}\right).$$

In Lemma 1.6, we choose $\lambda(z) \equiv \alpha'$, $\alpha = 1$, $\beta = 2(1+\varepsilon)/3$ and $\eta = \alpha'\beta$. Then (2.3) follows from Lemma 1.6 and therefore the proof of the theorem is complete.

For $\lambda(z) = 1$ and $\eta = \beta$ in Lemma 1.6 the following simple observation will be useful (see also [2]): if $f \in \mathcal{A}$ then

$$|\arg f'(z)| < \frac{(1+\varepsilon)\pi}{3} \text{ implies } \left|\arg\frac{f(z)}{z}\right| < \frac{\pi\beta}{2}, \tag{2.4}$$

where $0 < \beta \leq \beta_0$ is given by

$$\frac{2(1+\varepsilon)}{3} = \beta + \frac{2}{\pi}\arctan\beta, \ (-1 < \varepsilon \leq 2),$$

and β_0 is the solution of

$$y\pi = \frac{3\pi}{2} - \arctan y.$$

In particular, we have

$$\mathrm{Re}\ f'(z) > 0 \ \text{ implies } \ \left|\arg \frac{f(z)}{z}\right| < \frac{\pi\beta}{2}, \tag{2.5}$$

where $\beta \approx 0.63$ in (2.5) is the solution of $x = \cot(x\pi/2)$.

Example 2.6. From Lemma 1.6, it is simple to show that if $f \in \mathcal{A}$ satisfies

$$\mathrm{Re}\,(f'(z) + (\sqrt{3}/2)zf''(z)) > 0$$

then we have

$$|\arg f(z)| < \frac{\pi}{3}.$$

This suggests that there may exist a $\delta > 0$ and an ε with $1/3 < \varepsilon \leq 1/2$ for which

$$\mathrm{Re}\,(f'(z) + \alpha zf''(z)) > -\delta \ \text{ implies } \ |\arg f'(z)| < \varepsilon\pi,$$

for some $\alpha < \sqrt{3}/2$. A general solution to this problem will be presented later.

Choose $\alpha = 1/2$ in Theorem 2.1. Therefore, for $1/3 < \varepsilon \leq 1/2$ satisfying the condition that $\varepsilon \tan(\varepsilon\pi) \geq 1$, we have

$$\mathrm{Re}\left(f'(z) + \frac{1}{2}zf''(z)\right) > -\frac{[1 - 1/(\varepsilon\tan(\varepsilon\pi))][3 - 4\log 2]}{1 - [1 - 1/(\varepsilon\tan(\varepsilon\pi))][3 - 4\log 2]}$$

which implies

$$|\arg f'(z)| < \varepsilon\pi.$$

Theorem 2.7. *Let $\varepsilon \approx 0.3059$ be the solution of*

$$\frac{(1+\varepsilon)\pi}{3} = \beta + \arctan\left(\frac{2\beta}{\pi}\right) \tag{2.8}$$

with

$$\beta = \arctan\left(\frac{\sqrt{3}\alpha'}{1-\alpha'}\right)$$

and

$$\alpha' = \frac{3\cot((1+\varepsilon)\pi/3)}{2(1+\varepsilon)}.$$

If $f \in \mathcal{A}$ satisfies

$$\mathrm{Re}\,(f'(z) + \alpha zf''(z)) > -\frac{(1-\alpha'/\alpha)(2\rho(\alpha)-1)}{1-(1-\alpha'/\alpha)(2\rho(\alpha)-1)}$$

for $\alpha \geq \alpha' \approx 0.4036$, then $f \in S^*$.

Proof. By Lemma 1.5, it is sufficient to prove the theorem for $\alpha = \alpha'$. Suppose that f satisfies

$$\operatorname{Re}\left(f'(z) + \alpha' z f''(z)\right) > 0. \tag{2.9}$$

Then, by Theorem 2.1, we obtain

$$|\arg f'(z)| < \frac{(1+\varepsilon)\pi}{3}$$

which, by (2.4), implies that

$$\left|\arg \frac{f(z)}{z}\right| < \arctan\left(\frac{\sqrt{3}\alpha'}{1-\alpha'}\right) \tag{2.10}$$

with

$$\frac{(1+\varepsilon)\pi}{3} = \arctan\left(\frac{\sqrt{3}\alpha'}{1-\alpha'}\right) + \arctan\left\{\frac{2}{\pi}\arctan\left(\frac{\sqrt{3}\alpha'}{1-\alpha'}\right)\right\}, \tag{2.11}$$

which is in fact equivalent to (2.8).

Now, if we let $p(z) = zf'(z)/f(z)$ then p is analytic in Δ with $p(0) = 1$. Further, by arithmetic calculations, (2.9) can be written as

$$\psi(p(z), zp'(z); z) \in \Omega = \{\omega \in \mathbb{C} : \operatorname{Re}\omega > 0\}$$

where

$$\psi(r, s; z) = \alpha' \frac{f(z)}{z}\left[r^2 + \left(\frac{1}{\alpha'} - 1\right) r + s\right].$$

We wish to show that $\operatorname{Re} p(z) > 0$ in Δ which is equivalent to $f \in \mathcal{S}^*$. If we let $f(z)/z = U + iV$ then we can easily find that

$$\operatorname{Re}\psi(ix, y; z) = \alpha'[(y - x^2)U - (1/\alpha' - 1)Vx]$$

and, therefore, we deduce that

$$\operatorname{Re}\psi(ix, y; z) \le -\frac{\alpha'}{2}\left[3Ux^2 + 2\left(\frac{1}{\alpha'} - 1\right)Vx + U\right] \equiv Q(x)$$

for $y \le -(1+x^2)/2$ and x real. Since $f(z)/z = U + iV$, we may rewrite (2.10) as

$$|V| < \left(\frac{\sqrt{3}\alpha'}{1-\alpha'}\right) U,$$

and so the discriminant of $Q(x)$ is non-positive. This implies that $\operatorname{Re}\psi(ix, y; z) \le 0$ for $z \in \Delta$, x real and $y \le -(1+x^2)/2$. Hence, by Lemma 1.7 and Equation (1.8), we conclude that

$$\operatorname{Re} p(z) = \operatorname{Re}\left(zf'(z)/f(z)\right) > 0$$

and, therefore, f is starlike. This completes the proof of the Theorem 2.7.

As an immediate application of Lemma 1.6 we have the following:

Corollary 2.12. *If λ is a function defined on Δ such that*

$$|\mathrm{Im}\,\lambda(z)| \leq \tan\left(\frac{\pi\gamma}{2}\right)\left[\mathrm{Re}\,\lambda(z) - \frac{\cot(\pi\gamma/2)}{\gamma}\right],$$

for $0<\gamma<1$, then for p analytic in Δ with $p(0)=1$

$$\mathrm{Re}\,(p(z)+\lambda(z)zp'(z)) > 0 \quad \textit{implies} \quad |\arg p(z)| < \frac{\pi\gamma}{2}.$$

For our next result we generalize Corollary 2.12 as follows:

Theorem 2.13. *Let λ be a function defined on Δ satisfying*

$$|\mathrm{Im}\,\lambda(z)| < \tan\left(\frac{\pi\gamma}{2}\right)[\mathrm{Re}\,\lambda(z)-\eta], \quad (\eta>0) \tag{2.14}$$

for $0<\gamma<1$. If p is analytic in Δ with $p(0)=1$ and

$$\mathrm{Re}\,(p(z)+\lambda(z)zp'(z)) > \beta(\gamma,\eta), \tag{2.15}$$

where

$$\beta(\gamma,\eta) = t_0^{\gamma}\cos\left(\frac{\pi\gamma}{2}\right)\left[1-\frac{\gamma\eta}{2}\left(t_0+\frac{1}{t_0}\right)\tan\left(\frac{\pi\gamma}{2}\right)\right] \tag{2.16}$$

with

$$t_0 = \frac{1}{\eta(1+\gamma)}\left[\cot\left(\frac{\pi\gamma}{2}\right)\left\{1+\sqrt{1+\eta^2(1-\gamma^2)\tan^2(\pi\gamma/2)}\right\}\right],$$

then

$$|\arg p(z)| < \frac{\pi\gamma}{2}. \tag{2.17}$$

The following corollary is an elementary consequence of the above theorem and so we omit it derivation.

Corollary 2.18. *Let α be a complex number satisfying the condition*

$$\mathrm{Re}\,\alpha > |\mathrm{Im}\,\alpha|\cot(\pi\gamma/2) \quad \textit{with } 0<\gamma<1.$$

If p is analytic in Δ, $p(0)=1$ and

$$\mathrm{Re}\,(p(z)+\alpha zp'(z)) > \beta(\gamma, \mathrm{Re}\,\alpha - |\mathrm{Im}\,\alpha|\cot(\pi\gamma/2)),$$

where β *is as in Theorem* 2.13, *then*

$$|\arg p(z)| < \frac{\pi\gamma}{2}.$$

Remark 2.19. It can be easily shown that $\beta(\gamma,\eta) = 0$ when $\eta = [\cot(\pi\gamma/2)]/\gamma$. Therefore, Theorem 2.13 is an extension of Corollary 2.12. The case $\gamma = 2/3$ has already been considered by the author in a different context and further, this particular case has been used to obtain a new sufficient condition for a function $f \in \mathcal{A}$ to be starlike in Δ; see [6] for details. However, it will be interesting if one can use Corollary 2.18 (with $p(z) = f'(z)$) and the method of proof used in Theorem 2.7 to determine the correct range of the complex constant α for which $\mathcal{R}(\alpha, 0) \subset \mathcal{S}^*$. In [4] the author determined simple conditions on $\alpha \in \mathbb{C}$ and $\beta < 1$ for which

$$f \in \mathcal{A} \text{ and } \operatorname{Re}(f'(z) + \alpha z f''(z)) > \beta \text{ implies } f \in \mathcal{S}^*.$$

Remark 2.20. Let M and N be analytic in Δ with $M(0) = N(0) = 0$, and $M'(0)/N'(0) = 1$. Further, suppose that N maps Δ onto a multisheeted domain with respect to the origin. Now we consider

$$\lambda(z) = \frac{\alpha N(z)}{zN'(z)} \text{ and } p(z) = \frac{M(z)}{N(z)} \quad (\alpha > 0).$$

Then λ and p are analytic in Δ, $\operatorname{Re}\lambda(z) > 0$ in Δ, $p(0) = 1$, $\lambda(0) = \alpha$ and

$$p(z) + \lambda(z)zp'(z) = (1-\alpha)\frac{M(z)}{N(z)} + \alpha\frac{M'(z)}{N'(z)}.$$

If we allow $\gamma \to 1^-$ in Corollary 2.12, then, because of the above observation, we easily obtain

$$\operatorname{Re}\left((1-\alpha)\frac{M(z)}{N(z)} + \alpha\frac{M'(z)}{N'(z)}\right) > 0 \text{ implies } \operatorname{Re}\left(\frac{M(z)}{N(z)}\right) > 0.$$

This is a well-known result of Libera-Sakaguchi-Chichra (see for example [4]). This particular result has a number of applications and hence we expect that Theorem 2.13 may prove to be a useful tool in obtaining new results as well as in improving many of the known results available in the literature.

Proof of Theorem 2.13. Let us write (2.21)

$$q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}, \quad 0 < \gamma < 1.$$

Then we see that the function q is convex in Δ and $q(0) = 1$. For convenience, we set

$$H(z) = p(z) + \lambda(z)zp'(z).$$

Now we will use a well-known result of Miller and Mocanu [2] (For a more general form we refer to [1]). Suppose that p is not subordinate to q. According to Miller and Mocanu's result, there are points $z_0 \in \Delta$ and $\zeta_0 \in \partial\Delta$, and $m \geq 1$ such that

$$H(z_0) = p(z_0) + \lambda(z_0)z_0 p'(z_0) = q(\zeta_0) + m\lambda(z_0)\zeta_0 q'(\zeta_0). \tag{2.22}$$

If we let $\lambda(z_0) = U + iV$ ($\equiv \operatorname{Re}\lambda(z_0) + i\operatorname{Im}\lambda(z_0)$), then from (2.14) we, in particular, find that

$$\left.\begin{matrix} U + V\cot(\pi\gamma/2) \\ U - V\cot(\pi\gamma/2) \end{matrix}\right\} \geq U - |V|\cot(\pi\gamma/2) \geq \eta > 0. \tag{2.23}$$

Using (2.23), (2.22) becomes

$$H(z_0) = q(\zeta_0) + m(U + iV)\zeta_0 q'(\zeta_0). \tag{2.24}$$

Now we need to consider separately the case $p(z_0) \neq 0$ which corresponds to a point on one of the rays on the sector $q(\Delta)$, and the case $p(z_0) = 0$ which corresponds to the corner of the sector. Observe that the latter case occurs only when $\gamma \geq 1$. Since $0 < \gamma < 1$, it is sufficient to consider the case $p(z_0) \neq 0$. In this case we note that $\zeta_0 \neq \pm 1$. Therefore, if we let $ix = (1 + \zeta_0)/(1 - \zeta_0)$ and use (2.22) and (2.24), we obtain that

$$H(z_0) = (ix)^\gamma \left[1 + im\gamma(U + iV)\left(\frac{1 + x^2}{2x}\right)\right],$$

where

$$ix = \begin{cases} xe^{i\pi/2} & \text{if } x > 0 \\ -xe^{-i\pi/2} & \text{if } x < 0. \end{cases}$$

Therefore, we have

$$\operatorname{Re} H(z_0) = \begin{cases} |x|^\gamma \cos(\frac{\pi\gamma}{2})\left[1 - \frac{m\gamma}{2}\left(|x|^{-1} + |x|\right)\left(U\tan(\frac{\pi\gamma}{2}) + V\right)\right] & \text{if } x > 0 \\ |x|^\gamma \cos(\frac{\pi\gamma}{2})\left[1 - \frac{m\gamma}{2}\left(|x|^{-1} + |x|\right)\left(U\tan(\frac{\pi\gamma}{2}) - V\right)\right] & \text{if } x < 0. \end{cases}$$

Thus, from (2.23) and the fact that $\eta > 0$ and $m \geq 1$, the above identity reduces to

$$\operatorname{Re} H(z_0) \leq |x|^\gamma \cos(\frac{\pi\gamma}{2})\left[1 - \frac{\eta\gamma}{2}\left(|x|^{-1} + |x|\right)\tan(\frac{\pi\gamma}{2})\right] \equiv g(|x|), \text{ say,}$$

for $x \neq 0$, where

$$g(t) = t^\gamma \cos(\frac{\pi\gamma}{2})\left[1 - \frac{\eta\gamma}{2}\left(t^{-1} + t\right)\tan(\frac{\pi\gamma}{2})\right], \quad t = |x| > 0.$$

Since

$$t_0 = \frac{1}{\eta(1+\gamma)}\left[\cot\left(\frac{\pi\gamma}{2}\right)\left\{1 + \sqrt{1 + \eta^2(1 - \gamma^2)\tan^2(\pi\gamma/2)}\right\}\right]$$

is the point of maximum for $g(t)$, it follows that for z_0 in Δ,

$$\operatorname{Re} H(z_0) \leq g(|x|) \leq g(t_0) \equiv \beta(\gamma, \eta),$$

where $\beta(\gamma, \eta)$ is defined by (2.16). The above inequality contradicts (2.15) at $z = z_0 \in \Delta$ and hence we conclude that $p(z)$ is subordinate to $q(z) = ((1+z)/(1-z))^\gamma$ with $0 < \gamma < 1$.

We conclude with the following remark: This paper is a part of an internal report of School of Mathematics, **SPIC** Science Foundation from March 91 and the work was supported by NBHM. After the revised version of the paper was sent for publication, the author learnt that the inclusion $\mathcal{R}(\alpha, \beta(0, \alpha, 1/3)) \subset \mathcal{S}^*$ holds for $\alpha \geq 1/3$ and is an improvement of Theorem 2.7. This result has been obtained in a recent paper by R. Fournier and St. Ruscheweyh [On two extremal problems related to univalent functions, *Rocky Mountain J. Math.* **24**(2)(1994), 529-538] and is seem to be the better lower bound for the problem of Mocanu.

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