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PRIMARY ELEMENTS AND PRIME POWER ELEMENTS IN MULTIPLICATIVE LATTICES

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Throughout, we assume that L is a C-lattice. For any prime element p of L, L_p denotes the localization at $\{x \in C | x \not\leq p\}$, where C is a multiplicative closed subset of compact elements of L which generates L under joint. For details see [7].

We shall begin with the following definitions.

Definition 1. L is said to satisfy the conditon (α) if every primary element is a power of its radical.

Definition 2. L is said to satisfy the condition (δ) if every element is a finite meet of prime power elements.

Let R be a commutative ring with identity. If every primary ideal of R is a power of its radical, then $L(R)$ (the lattice of ideals of R) is an r-lattice satisfying the conditon (α). If every ideal of R is a finite intersection of prime power ideals, then $L(R)$ is an r-lattice whcih satisfies the condition (δ). If L is a principally generated M-lattice (for definition see [6]), then L satisfies the condition (α) (see Lemma 4.4 of [2]). If L is a principal element lattice, then L satisfies the condition (δ) (see Theorem 5 of [8]).

Craig A. Wood, H. S. Butts and R. W. Gilmer have studied these conditions in the case of commutative rings (see [3] and [4]).

We need some more definitions to prove the main results.

Defintion 3. A prime element p of L is said to be an α-prime if every primary element $q \leq p$, is a power of its radical.

Definition 4. A prime element p of L is called a weak δ-prime if every element $a \leq p$ is a finite meet of prime power elements.

Definition 5. A prime element p of L is called a δ-prime if every element $a \leq p$ is a finite meet of powers of α-prime elements.

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Note that L satisfies the condition (α) if and only if every prime is an α-prime. Obviously, every δ-prime element is a weak δ-prime element.

We prove some lemmas that we need.

Lemma 1. *If p is a weak δ-prime element, then p is an α-prime element.*

Proof. The proof of the lemma is similar to the proof of Theorem 8 of [3].

Lemma 2. *The following statements on L are equivalent:*
(i) *L satisfies the conditon (δ).*
(ii) *Every prime element of L is a weak δ-prime element.*
(iii) *Every prime element is a δ-prime element*

Proof. (i) $\Leftrightarrow$(ii) directly follows from the defintions and (iii) $\Rightarrow$ (ii) is obvious. (ii) $\Rightarrow$ (iii) follows from Lemma 1.

If $\{p_\alpha\}$ is the collection of prime elements minimal over a, then by the isolated primary component of a belonging to p_α (or the isolated p_α-primary component of a) we mean the meet $\wedge q_\alpha$ of all p_α-primary elements which contain a.

Note that in L , every finite product of compact elements is compact. Therefore if $a \leq p$ and p is a prime element, then p is a minimal prime over a if and only if for any compact element $x \in L$, $x \leq p$ implies there exists a compact element $y \not\leq p$ such that $x^n y \leq a$ for some positive integer n (see Lemma 3.5 of [2]). Further if p is a minimal prime over a, then the isolated p-primary component of a is a p-primary element (see Lemma 3.8 of [2]).

Lemma 3. *Let p be a minimal prime over a $(a,p \in L)$ and let q be the isolated p-primary component of a. Then*

$$q = \bigvee\{x \in C | xy \leq a, y \not\leq p \quad for\ some \quad y \in C\} = a_p$$

Proof. The proof of the lemma is straightforward and hence is omitted.

Lemma 4. *Let p be a prime element of L. For any $n \in \mathbb{Z}^+$. let $p^{(n)} = \bigvee\{x \in C \mid xy \leq p^n, y \not\leq p$ for some $y \in C\}$. Then $p^{(n)}$ is the isolated primary component of p^n.*

Proof. The proof of the lemma follows from Lemma 3.

For each $n \in \mathbb{Z}^+$, $p^{(n)}$ is called the *nth* symbolic power of p. Note that $p^{(n)} = p^n_p$ and $p^{(n)} \circ_p p^{(k)} = p^n_p \circ p^k_p = (p^n p^k)_p = p^{n+k}_p = p^{(n+k)}$.

Lemma 5. *Let p be a prime element of L and let every p-primary element be a power of p. If the symbolic powers of p properly descend, then for each $n \in \mathbb{Z}^+$, $p^{(n)} = p^n$. Hence each p^n is p-primary.*

Proof. The proof follows by induction on n.

For each prime element $p \in L$, Let $p^w = \bigwedge_{n=1}^{\infty} p^n$ and $p^{(w)} = \bigwedge_{n=1}^{\infty} p^{(n)}$.

Lemma 6. *Let p be a prime element of L. If $p^{(w)}$ is a prime element and if $p^{(w)} < p$, then the symbolic powers of p properly descend.*

Proof. Observe that $p^{(w)} = \bigwedge_{n=1}^{\infty} p^{(n)} = \bigwedge_{n=1}^{\infty} p_p^n$, so $p^{(w)} = p_p^{(w)} = (\bigwedge_{n=1}^{\infty} p_p^n)_p = \bigwedge_p p_p^n = \bigwedge_p p^{(n)}$. If $p^{(n)} = p^{(n+1)} = p^{(n)} \circ_p p^{(1)}$, then $p^{(n)} = p^{(k)}$ for all $k \geq n$, so $p^{(w)} = p^{(n)}$. Since $p^n \leq p^{(n)} = p^{(w)}$ and $p^{(w)}$ is a prime element, it follows that $p \leq p^{(w)}$ which is a contradiction. Therefore the symbolic powers of p properly descend. This completes the proof the lemma.

An element $a \in L$ is said to be a strong join principal element if a is join principal and compact. An element $b \in L$ is called prime to a $(a < 1)$ if whenever $bc \leq a$, then $c \leq a$.

Lemma 7. *Let d be a strong join principal element of L and let d be prime to an element $b \in L$. Suppose p is a prime element minimal over $d \vee b$ and let q_i be the isolated p-primary component of $d^i \vee b$. Then $q_1 > q_2 > q_3 > \cdots$.*

Proof. Clearly $q_1 \geq q_2 \geq \cdots$. We show that for each i, $d^i \leq q_i$ and $d^i \not\leq q_{i+1}$. Obviously $d^i \leq q_i$. If $d^i \leq q_{i+1}$, then by Lemma 3, $d^i y \leq d^{i+1} \vee b = d^i d \vee b$ for some compact element $y \not\leq p$. Since d^i is prime to b, it follows that $(b : d^i) \leq b$. As $y \leq (d^i d \vee b : d^i)$ and d^i is join principal, we get $y \leq d \vee (b : d^i) \leq d \vee b \leq p$ which is a contradiction. Therefore $d^i \not\leq q_{i+1}$ and hence $q_1 > q_2 > q_3 > \cdots$.

Theorem 1. *Let p be a prime element of L and let every p-primary element be a power of its radical. Let d be a strong join principal element and let d be prime to an element $b \in L$. If p is minimal prime over $d \vee b$, then the powers of p, properly descend, $b \leq p^w$ and p^w is the meet of all p-primary elements of L.*

Proof. The proof of the theorem is simsilar to the proof of Theorem 1 of [3].

Theorem 2. *Suppose L is generated by strong join principal elements. Let m be an α-prime element and let p_0 be a prime element such that $p_0 < m$. Then $p_0 \leq m^w$.*

Proof. Choose any strong join principal element d such that $d \leq m$ and $d \not\leq p_0$. Then d is prime to p_0. Let $p \leq m$ be a minimal prime over $d \vee p_0$. Then by Theorem 1, $p_0 \leq p^w \leq m^w$. This completes the proof of the theorem.

Lemma 8. *Suppose L is generated by strong join principal elements. Let p and m be prime elements such that $p < m$ and there are no prime elements strictly between p and m. If every m-primary element is a power of m, then*

$p = \bigwedge_{k=1}^{\infty} m^{(k)} = \bigwedge_{k=1}^{\infty} m^k$.

Proof. We show that $\bigwedge_{k=1}^{\infty} m^{(k)} \leq p$. Let x be any strong join principal element such that $x \leq \bigwedge_{k=1}^{\infty} m^{(k)}$ and $x \not\leq p$. Then $x^2 \not\leq p$, so m is a minimal prime over $p \vee x^2$. Let $q = \bigvee\{y \in C \mid yz \leq x^2 \vee p, z \not\leq m$, for some $z \in C\}$. The q is m-primary, so by hypothesis, $q = m^k$ for some $k \in \mathbb{Z}^+$. As m^k is m-primary, it follows that $m^k = m^{(k)}$ and hence $x \leq q$. As x is compact, $xa \leq x^2 \vee p$ for some $a \not\leq m$. As x is join principal, $a \leq (x^2 \vee p : x) = x \vee (p : x) \leq x \vee p$ since $(p : x) \leq p$. Therefore $a \leq p \vee x \leq m$, a contradiction and hence $\bigwedge_{k=1}^{\infty} m^{(k)} \leq p$. The remaining part follows from Theorem 1.

Theorem 3. *Let L be generated by strong join principal elements. Let m be a nonminimal prime element. If m is an α-prime, then $m^w = \bigwedge_{k=1}^{\infty} m^k$ is a prime element containing each prime element properly contained in m. Further each m^k is primary.*

Proof. If $m = m^2$, the we are through. So assume that $m^2 < m$. Let $\mathcal{Y} = \{p \in L \mid p$ is prime and $p < m\}$. By hypothesis $\mathcal{Y} \neq \emptyset$. By Theorem 2 and by Zorn's lemma, $\mathcal{Y}$ contains a maximal element p such that p is prime, $p < m$ and there are no prime elements properly between p and m. By Lemma 8, $p = m^w = m^{(w)}$. By Theorem 2, p contains each prime element properly contained in m. Further by Lemma 5 and Lemma 6, each m^k is m-primary. This completes the proof of the theorem.

As consequences, we have the following results.

Corollary 1. *Let L be generated by strong join principal elements. If L satisfies the condition (α), then $p^n (n \in \mathbb{Z}^+)$ is p-primary for every nonminimal prime element p of L.*

Corollary 2. *Let L be generated by strong join principal elements. If L is a domain and if p is an α-prime, then p^n is p-primary for each $n \in \mathbb{Z}^+$.*

Corollary 3. *Let L be generated by strong join principal elements. If L is a domain and if L satisfied the condition (α), then prime power elements are primary.*

Lemma 9. *Suppose L is generated by strong join principal elements. Let $\sqrt{a} = p$ and p be a δ-prime element. Then $a = p^n$ for some $n \in \mathbb{Z}^+$.*

Proof. By using Theorem 2 and by imitating the proof of Theorem 9 of [3], we can get the result.

Lemma 10. *Suppose L is generated by strong join principal elements. If p is a nonminimal δ-prime element, then p is maximal.*

Proof. Suppose p is a nonminimal δ-prime element. Then $p_1 < p$ for some prime element $p_1 \leq L$. Choose any strong join principal element $d \leq p$ such that $d \not\leq p$. Let $p_0 \leq p$ be a minimal prime over $d \vee p_1$. Note that p_0 is a δ-prime and hence it is an α-prime. Again by Lemma 7, $p_0 \neq p_0^2$. Choose any strong join principal $y \leq p_0$ such that $y \not\leq p_0^2$. Suppose p is nonmaximal. Then $p < m$ for some maximal element m of L. Since by Theorem 3, p_0^2 is p_0-primary, $my \not\leq p_0^2$, so $p_0^2 < p_0^2 \vee my \leq p_0$ and therefore $\sqrt{p_0^2 \vee my} = p_0$. By Lemma 9, $p_0^2 \vee my = p_0$, so $y \leq my \vee p_0^2$ and hence $1 = (my \vee p_0^2 : y)$. Again since y is join principal, we have $1 = m \vee (p_0^2 : y) \leq m \vee p_0$ (since $(p_0^2 : y) \leq p_0) = m$, a contradiction. Therefore p is maximal and this completes the proof of the lemma.

For the definitons of discrete valuation lattices and special principa element lattices, the reader is referred to [10].

Lemma 11. *Suppose L is principally generated. If m is a nonminimal δ-prime element, then L_m is a one -dimensional discrete valuation lattice.*

Proof. Note that by Lemma 10, $\dim L_m = 1$, so every element has a prime radical. Again as m is a δ-prime, by Lemma 9, every element of L_m is a prime power. Consequently L_m is totally ordered. Again by Lemma 4.8 of [1], L_m is a one-dimensional discrete valuation lattice.

Lemma 12. *Suppose L is principally generated. If m is a δ-prime element whcih is both maximal and minimal, then L_m is a special principal element lattice.*

Proof. The proof of the lemma follows from Lemma 9.

Theorem 4. *Suppose L is principally generated. Then the following statements are equivalent:*
(i) *L satisfies the condition (δ).*
(ii) *Every prime element is a weak δ-prime element.*
(iii) *Every prime element is a δ-prime element.*
(iv) *Every element is principal.*

Proof. By Lemma 2, (i), (ii), and (iii) are equivalent, (iii) $\Rightarrow$ (iv). Suppose (iii) holds. By Lemma 10, $\dim L \leq 1$. Next we show that every prime element is weak meet principal. Let m be a prime element. Suppose $a \leq m$. By hypothesis, $a = p_1^{\alpha_1} \wedge \cdots \wedge p_n^{\alpha_n}$ for some prime elements $p_i \in L (i = 1, 2, \ldots, n)$. Without loss of generality, assume that p_i's are distinct.

We can also assume that $p_i \not\leq p_j$ for $i \neq j$ $(1 \leq i, j \leq n)$. By Lemma 1, every prime element is an α-prime. As $\dim L \leq 1$, by Theorem 3, p_i's are comaximal and hence $a = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_n^{\alpha_n}$. As $a \leq m$, $p_i \leq m$ for some i. If $p_i = m$, then we are through. Suppose $p_i < m$. By Lemma 11, $p_{i_m} = 0_m$ and so $ab = 0$ for some $b \not\leq m$. Since m is maximal, it follows that $a = am$. This shows that every prime element is weak meet principal. Again note that L contains only a finite number of minimal primes and hence by Theorem 1.5 of [9], L is a principal element lattice. (iv) $\Rightarrow$ (iii) follows from Theorem 5 of [8]. This completes the proof of the theorem.

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