TAMKANG JOURNAL OF MATHEMATICS
Volume 27, Number 2, Summer 1996

A MANY VARIABLE GENERALIZATION OF THE DISCRETE HARDY'S INEQUALITY

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Abstract. In the present note we establish the discrete Hardy's inequalities in many variables. The main tools used for deriving the inequalities are base on the Fubini's theorem and some application of the fundamental inequalities.

1. Introduction

In [2] G. H. Hardy established the following inequality concerning a series of terms.

Theorem A. *If* $P > 1, a_n \geq 0$ *and* $A_n = a_1 + a_2 + \cdots + a_n$ *then*

$$\sum \left(\frac{A_n}{n}\right)^P \leq \left(\frac{P}{P-1}\right)^P \sum a_n^P. \tag{1}$$

The equality holds in (1) *if all the a are zero.*

In [1] Copson established the following generalization of the inequality (1)

Theorem B. *If* $P > 1, \lambda_n > 0, a_n > 0, \Lambda_n = \lambda_1 + \lambda_2 + \cdots + \lambda_n, A_n = \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n$ *and* $\sum \lambda_n a_n^P$ *converges, then*

$$\sum \lambda_n \left(\frac{A_n}{\Lambda_n}\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum \lambda_n a_n^p \tag{2}$$

The constant is the best possible.

In [4] B. G. Pachpatte established the following generalizatoin of the inequality (2).

Theorem C. *Let* $p, \lambda_n, a_n, \Lambda_n$ *and* A_n *be as in Theorem B and let* $H(u)$ *be a real-valued positive convex function defined for* $u > 0$.

Received October 12, 1994.
1991 *Mathematics Subject Classification.* Primary 26D15.
Key words and phrases. Discrete Hardy's inequality.

If $\sum \lambda_n H^p(a_n)$ *converges, then*

$$\sum \lambda_n H^p\Big(\frac{A_n}{\Lambda}\Big) \leq \Big(\frac{p}{p-1}\Big)^p \sum \lambda_n H^p(a_n). \tag{3}$$

The constant is the best possible.

In [3] Dah-Yan Hwang and Gou-Sheng Yang established the following inequalities which generalize (2) and (3).

Theorem D. *Let* $p > 1, \beta_n > 0, \lambda_n > 0, a_n > 0, \sum \lambda_n a_n^p$ *converge, and further let* $\Lambda_n = \sum_{i=1}^n \beta_i \lambda_i, A_n = \sum_{i=1}^n \beta_i \lambda_i a_i,$
If there exists $K > 0$ *such that*

$$P - 1 + \frac{(\beta_{n+1} - \beta_n)\Lambda_n}{\beta_{n+1} \cdot \beta_n \cdot \lambda_n} \geq \frac{P}{K} \quad for \quad n = 1, 2, 3, \ldots$$

then

$$\sum \lambda_n \Big(\frac{A_n}{\Lambda}\Big)^p \leq K^p \sum \lambda_n a_n^p \tag{4}$$

The case $\beta_n = 1, n = 1, 2, 3, \ldots$ *and* $K = \frac{P}{P-1}$ *shows the constant in* (4) *to be the best possible.*

Theorem E. *Let* H *be as in Theorem C, and let* $P, \beta_n, \lambda_n, a_n, \Lambda_n, A_n$ *and* K *be as in Theorem D. If* $\sum \lambda_n H^p(a_n)$ *converges, then*

$$\sum \lambda_n H^p\Big(\frac{A_n}{\Lambda_n}\Big) \leq K^p \sum \lambda_n H^p(a_n). \tag{5}$$

Theorem F. *Let* $P, \beta_n . \lambda_n, a_n, \Lambda_n, A_n$ *and* K *be as in Theorem D. and let* $\varphi > 0$ *be defined on* $(0, \infty)$ *so that* $\varphi > 0$ *and* $\varphi \cdot \varphi'' \geq \Big(1 - \frac{1}{P}\Big)(\varphi')^2$ *If* $\sum \lambda_n \varphi(a_n)$ *converges, then*

$$\sum \lambda_n \varphi\Big(\frac{A_n}{\Lambda}\Big) \leq K^P \sum \lambda_n \varphi(a_n). \tag{6}$$

In [5] B. G. Pachpatte has recently established a many variable generalization of the inequality (1).

In what follows, we let R be the set of real numbers and B be a subset of the n-dimensional Euclidean space R^n defined by $B = \{x \in R^n : e \leq x < \infty\}$ where $e = (1, 1, \ldots, 1) \in R^n$. For a function $u : B \to R$, we use the following notations

$$\sum_B u(y) = \sum_{y_1=1}^{\infty} \cdots \sum_{y_n=1}^{\infty} u(y_1, y_2, \ldots y_n)$$

and

$$\sum_{B(e,x)} u(y) = \sum_{y_1=1}^{x_1} \cdots \sum_{y_n=1}^{x_n} u(y_1, y_2, \dots y_n)$$

where $e = (1, \dots, 1) \in B, x = (x_1, x_2, \dots, x_n) \in B$ such that $e \leq x$, i.e. $1 \leq x_i$

His main result is established in the following theorem.

Theorem G. *If $P > 1$ is a constant, $f(x) \geq 0$ for $x \in B$ and $A(x) = \sum_{B(e,x)} f(x), x \in B$, then*

$$\sum_B \Big\{\frac{A(x)}{\pi_{i=1}^n x_i}\Big\}^p \leq \Big(\frac{p}{p-1}\Big)^{np} \sum_B f^p(x). \tag{7}$$

The equality holds in (7) if $f(x) = 0$ for all $x_i, i = 1, \dots, n$.

In the present note we will establish some new inequalities whcih generalize the inequalities (4), (5), (6), and (7).

2. Main results

Theorem 1. *If $P > 1, a(x) > 0, \beta_i(x_i) > 0, \lambda_i(x_i) > 0$ for $x_i \geq 1, i = 1, 2, \dots, n$. and $\sum_B \pi_{i=1}^n \lambda_i(x_i) a^p(x), x \in B$ converges and further let $\Lambda_i(x_i) = \sum_{y_i=1}^{x_i} \beta_i(y_i)\lambda_i(y_i)$ for $i = 1, 2, \dots, n$ and $A(x) = \sum_{B(e,x)} \{\pi_{i=1}^n \beta_i(y_i)\lambda_i(y_i)\} a(y), x \in B$. If there exist $K_i > 0$ such that*

$$P - 1 + \frac{[\beta_i(x_i+1) - \beta_i(x_i)]\Lambda_i(x_i)}{\beta_i(x_i+1)\beta_i(x_i)\lambda_i(x_i)} \geq \frac{P}{K_i} \tag{8}$$

for $x_i \geq 1, i = 1, 2, \dots, n$ then

$$\sum_B \pi_{i=1}^n \lambda_i(x_i) \Big\{\frac{A(x)}{\pi_{i=1}^n \Lambda_i(x_i)}\Big\}^p \leq (\pi_{i=1}^n K_i)^p \sum_B \pi_{i=1}^n \lambda_i(x_i) a^p(x). \tag{9}$$

The equality holds in (9) if $a(x) = 0$ for all $x_i, i = 1, 2, \dots, n$.

Proof. Define

$$a_{n-j}(x_{n-j}) = \Big[\sum_{y_1=1}^{x_1} \cdots \sum_{y_{n-j}=1}^{x_{n-j}} \pi_{i=1}^{n-j} \beta_i(y_i)\lambda_i(y_i) a(y_1, \dots, y_{n-j}, x_{n-j+1}, \dots, x_n)\Big] \Lambda_{n-j}^{-1}(x_{n-j})$$

for $j = 0, 1, 2, \dots, n-1$, and $x_{n-j} \geq 1$, and let

$\alpha_i(0) = \lambda_i(0) = \beta_i(0) = 1$, for $i = 1, 2, \dots, n$

By making use of the elementary inequality

$$s^p + (p-1)t^p \geq pst^{p-1}.$$

where s and t are nonnegative numbers and agree that $\Lambda_i(0) = 0$ for $i = 1, 2, \ldots, n$ We have, for $x_n = 0, 1, 2, \ldots$

$$
\begin{aligned}
&- p\lambda_n(x_n+1)\sum_{y_1=1}^{x_1}\cdots\sum_{y_{n-1}=1}^{x_{n-1}} \pi_{i=1}^{n}\beta_i(y_i)\lambda_i(y_i)a(y_1,\cdots,y_{n-1},x_n+1)\alpha_n^{p-1}(x_n+1)\\
=&- p\beta_n(x_n+1)\lambda_n(x_n+1)\sum_{y_1=1}^{x_1}\cdots\sum_{y_{n-1}=1}^{x_{n-1}} \pi_{i=1}^{n}\beta_i(y_i)\lambda_i(y_i)a(y_1,\ldots,y_{n-1},x_n+1)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times\frac{\alpha_n^{p-1}(x_n+1)}{\beta_n(x_n+1))}\\
=&- p\{\alpha_n(x_n+1)\cdot\Lambda_n(x_n+1) - \alpha_n(x_n)\cdot\Lambda_n(x_n)\}\frac{\alpha_n^{p-1}(x_n+1)}{\beta_n(x_n+1))}\\
=&- p\Big[\frac{\Lambda_n(x_n+1)}{\beta_n(x_n+1)}\Big\}\alpha_n^p(x_n+1) + p\Big[\frac{\Lambda_n(x_n)}{\beta_n(x_n+1)}\Big]\alpha_n(x_n)\alpha_n^{p-1}(x_n+1)\\
\le&- p\Big[\frac{\Lambda_n(x_n+1)}{\beta_n(x_n+1)}\Big]\alpha_n^p(x_n+1)+\Big[\frac{\Lambda_n(x_n)}{\beta_n(x_n+1)}\Big]\alpha_n^p(x_n) + (p-1)\Big[\frac{\Lambda_n(x_n)}{\beta_n(x_n+1)}\Big]\alpha_n^p(x_n+1).
\end{aligned}
$$

so that

$$
\begin{aligned}
&(p-1)\lambda_n(x_n+1)\alpha_n^p(x_n+1) + \frac{[\beta_n(x_n+1)-\beta_n(x_n)]\Lambda_n(x_n)}{\beta_n(x_n+1)\beta_n(x_n)\lambda_n(x_n)}\lambda_n(x_n)\alpha_n^p(x_n)\\
&- p\lambda_n(x_n+1)\sum_{y_1=1}^{x_1}\cdots\sum_{y_{n-1}=1}^{x_{n-1}} \pi_{i=1}^{n-1}\beta_i(y_i)\lambda_i(y_i)a(y_1,\ldots,y_{n-1},x_n+1)\alpha_n^{p-1}(x_n+1) \quad (10)\\
\le&(p-1)\lambda_n(x_n+1)\alpha_n^p(x_n+1) + \frac{[\beta_n(x_n+1)-\beta_n(x_n)]\Lambda_n(x_n)}{\beta_n(x_n+1)\beta_n(x_n)\lambda_n(x_n)}\lambda_n(x_n)\alpha_n^p(x_n)\\
&- p\Big[\frac{\Lambda_n(x_n+1)}{\beta_n(x_n+1)}\Big]\alpha_n^p(x_n+1) + \Big[\frac{\Lambda_n(x_n)}{\beta_n(x_n+1)}\Big]\alpha_n^p(x_n) + (p-1)\Big[\frac{\Lambda_n(x_n)}{\beta_n(x_n+1)}\Big]\alpha_n^p(x_n+1)\\
=&(p-1)\Big[\frac{\Lambda_n(x_n+1)}{\beta_n(x_n+1)}\Big]\alpha_n^p(x_n+1) + \frac{\Lambda_n(x_n)}{\beta_n(x_n)}\alpha_n^p(x_n) - p\Big[\frac{\Lambda_n(x_n+1)}{\beta_n(x_n+1)}\Big]\alpha_n^p(x_n+1)\\
=&\frac{\Lambda_n(x_n)}{\beta_n(x_n)}\alpha_n^p(x_n) - \frac{\Lambda_n(x_n+1)}{\beta_n(x_n+1)}\alpha_n^p(x_n+1).
\end{aligned}
$$

By adding the inequalities for $x_n = 0, 1, 2, \ldots, N-1$, we have

$$
\begin{aligned}
&\sum_{x_n=0}^{N-1}(p-1)\lambda_n(x_n+1)\alpha_n^p(x_n+1) + \sum_{x_n=0}^{N-1}\frac{[\beta_n(x_n+1)-\beta_n(x_n)]\Lambda_n(x_n)}{\beta_n(x_n+1)\beta_n(x_n)\lambda_n(x_n)}\lambda_n(x_n)\alpha_n^p(x_n)\\
&- \sum_{x_n=0}^{N-1} p\lambda_n(x_n+1)\sum_{y_1=1}^{x_1}\cdots\sum_{y_{n-1}=1}^{x_{n-1}} \pi_{i=1}^{n-1}\beta_i(y_i)\lambda_i(y_i)a(y_1,\ldots,y_{n-1},x_n+1)\alpha_n^{p-1}(x_n+1)\\
\le&- \frac{\Lambda_n(N)}{\beta_n(N)}\alpha_n^p(N)\\
\le&0
\end{aligned}
$$

Thus,

$$\sum_{x_n=1}^{N}(p-1)\lambda_n(x_n)\alpha_n^p(x_n)+\sum_{x_n=0}^{N-1}\frac{[\beta_n(x_n+1)-\beta_n(x_n)]\Lambda_n(x_n)}{\beta_n(x_n+1)\beta_n(x_n)\lambda_n(x_n)}\lambda_n(x_n)\alpha_n^p(x_n)$$
$$\leq p\sum_{x_n=1}^{N}\lambda_n(x_n)\sum_{y_1=1}^{x_1}\sum_{y_2=1}^{x_2}\cdots\sum_{y_{n-1}=1}^{x_{n-1}}\pi_{i=1}^{n-1}\beta_i(y_i)\lambda_i(y_i)a(y_1,y_2,\ldots,y_{n-1},x_n)\alpha_n^{p-1}(x_n).$$

Using (8) and since $(p-1)\lambda_n(N)\alpha_n^p(N)\geq 0$, we have

$$\sum_{x_n=1}^{N-1}\lambda_n(x_n)\alpha_n^p(x_n)$$
$$\leq k_n\sum_{x_n=1}^{N}\lambda_n(x_n)\sum_{y_1=1}^{x_1}\cdots\sum_{y_{n-1}=1}^{x_{n-1}}\pi_{i=1}^{n-1}\beta_i(y_i)\lambda_i(y_i)a(y_1,\ldots,y_{n-1},x_n)\alpha_n^{p-1}(x_n). \tag{11}$$

Let N tend to infinity in (11) and using Hölder inequality with indices p and $\frac{p}{p-1}$ we have

$$\sum_{x_n=1}^{\infty}\lambda_n(x_n)\alpha_n^p(x_n)$$
$$\leq k_n\sum_{x_n=1}^{\infty}\lambda_n(x_n)\sum_{y_1=1}^{x_1}\cdots\sum_{y_{n-1}=1}^{x_{n-1}}\pi_{i=1}^{n-1}\beta_i(y_i)\lambda_i(y_i)a(y_1,\ldots,y_{n-1},x_n)\alpha_n^{p-1}(x_n)$$
$$=k_n\sum_{x_n=1}^{\infty}\Big[\lambda_n^{\frac{1}{p}}(x_n)\sum_{y_1=1}^{x_1}\cdots\sum_{y_{n-1}=1}^{x_{n-1}}\pi_{i=1}^{n-1}\beta_i(y_i)\lambda_i(y_i)a(y_1,\ldots,y_{n-1},x_n)\Big][\lambda_n^{\frac{p-1}{p}}(x_n)\alpha_n^{p-1}(x_n)]$$
$$\leq k_n\Big\{\sum_{x_n=1}^{\infty}\lambda_n(x_n)\Big[\sum_{y_1=1}^{x_1}\cdots\sum_{y_{n-1}=1}^{x_{n-1}}\pi_{i=1}^{n-1}\beta_i(y_i)\lambda_i(y_i)a(y_1,\ldots,y_{n-1},x_n)\Big]^p\Big\}^{\frac{1}{p}}$$
$$\times\Big\{\sum_{x_n=1}^{\infty}\lambda_n(x_n)\alpha_n^p(x_n)\Big\}^{\frac{p-1}{p}}$$

Dividing the above inequality by the last factor on the right and rasing the result to the pth power, we have

$$\sum_{x_n=1}^{\infty}\lambda_n(x_n)\alpha_n^p(x_n)$$
$$\leq k_n^p\sum_{x_n=1}^{\infty}\lambda_n(x_n)\Big[\sum_{y_1=1}^{x_1}\cdots\sum_{y_{n-1}=1}^{x_{n-1}}\pi_{i=1}^{n-1}\beta_i(y_i)\lambda_i(y_i)a(y_1,\ldots,y_{n-1},x_n)\Big]^p. \tag{12}$$

From (12) and using Fubini's theorem we have

$$\sum_B \pi_{i=1}^n \lambda_i(x_i)\Big[\frac{A(x)}{\pi_{i=1}^n \Lambda_i(x_i)}\Big]^p$$
$$\le \sum_{x_1=1}^{\infty}\cdots\sum_{x_{n-1}=1}^{\infty} \pi_{i=1}^{n-1}\lambda_i(x_i)[\pi_{i=1}^{n-1}\Lambda_i(x_i)]^{-p}k_n^p \sum_{x_n=1}^{\infty}\lambda_n(x_n)$$
$$\times\Big[\sum_{y_1=1}^{x_1}\cdots\sum_{y_{n-1}=1}^{x_{n-1}} \pi_{i=1}^{n-1}\beta_i(y_i)\lambda_i(y_i)a(y_1,\dots,y_{n-1},x_n)\Big]^p$$
$$=k_n^p\sum_{x_n=1}^{\infty}\lambda_n(x_n)\sum_{x_1=1}^{\infty}\cdots\sum_{x_{n-2}=1}^{\infty}\pi_{i=1}^{n-2}\lambda_i(x_i)[\pi_{i=1}^{n-2}\Lambda_i(x_i)]^{-p}$$
$$\times\sum_{x_{n-1}=1}^{\infty}\lambda_{n-1}(x_{x-1})\Lambda_{n-1}^{-p}(x_{n-1})\Big[\sum_{y_1=1}^{x_1}\cdots\sum_{y_{n-1}=1}^{x_{n-1}}\pi_{i=1}^{n-1}\beta_i(y_i)\lambda_i(y_i)a(y_1,\dots,y_{n-1},x_n)\Big]^p \tag{13}$$

Now by following exactly the same arguments as above we have,

$$\sum_{x_{n-1}=1}^{\infty}\lambda_{n-1}(x_{n-1})\Lambda_{n-1}^{-p}(x_{n-1})\Big[\sum_{y_1=1}^{x_1}\cdots\sum_{y_{n-1}=1}^{x_{n-1}}\pi_{i=1}^{n-1}\beta_i(y_i)\lambda_i(y_i)a(y_1,\dots,y_{n-1},x_n)\Big]^p$$
$$=\sum_{x_{n-1}=1}^{\infty}\lambda_{n-1}(x_{n-1})\alpha_{n-1}^p(x_{n-1})$$
$$\le k_{n-1}^p\sum_{x_{n-1}=1}^{\infty}\lambda_{n-1}(x_{n-1})\Big[\sum_{y_1=1}^{x_1}\cdots\sum_{y_{n-2}=1}^{x_{n-2}}\pi_{i=1}^{n-2}\beta_i(y_i)\lambda_i(y_i)a(y_1,\dots,y_{n-2},x_{n-1},x_n)\Big]^p. \tag{14}$$

From (13), (14) and using Fubini's theorem, we have

$$\sum_B \pi_{i=1}^n \lambda_i(x_i)\Big[\frac{A(x)}{\pi_{i=1}^n \Lambda_i(x_i)}\Big]^p$$
$$\le(k_{n-1}k_n)^p\sum_{x_{n-1}=1}^{\infty}\sum_{x_n=1}^{\infty}\lambda_{n-1}(x_{n-1})\lambda_n(x_n)\sum_{x_1=1}^{\infty}\cdots\sum_{x_{n-2}=1}^{\infty}\pi_{i=1}^{n-2}\lambda_i(x_i)[\pi_{i=1}^{n-2}\Lambda_i(x_i)]^{-p}$$
$$\times\Big[\sum_{y_1=1}^{x_1}\cdots\sum_{y_{n-2}=1}^{x_{n-2}}\pi_{i=1}^{n-2}\beta_i(y_i)\lambda_i(y_i)a(y_1,\dots,y_{n-2},x_{n-1},x_n)\Big]^p.$$

Continuing in this way, we finally get

$$\sum_B \pi_{i=1}^n \lambda_i(x_i)\Big[\frac{A(x)}{\pi_{i=1}^n \Lambda_i(x_i)}\Big]^p \le (\pi_{i=1}^n k_i)^p\sum_B \pi_{i=1}^n\lambda_i(x_i)a^p(x). \tag{15}$$

The proof of the theorem is complete.

Remark 1. Theorem 1 reduce to Theorem D when $n = 1$ and Theorem G when $k_i = \frac{p}{p-1}, \beta_i(x_i) = 1, \lambda_i(x_i) = 1$, for $i = 1, 2, \ldots, n$, $x_i \geq 1$.

Theorem 2. *Let H be a real-valued positive convex function defined on $(0, \infty)$ and let $p, a(x), \beta_i(x_i), \lambda_i(x_i), \Lambda_i(x_i), k_i$ for $i = 1, 2, \ldots, n$. and $A(x)$ be as in Theorem 1. If $\sum_B \pi_{i=1}^n \lambda_i(x_i) H^p(a(x))$ converges, then*

$$\sum_B \pi_{i=1}^n \lambda_i(x_i) H^p\Big(\frac{A(x)}{\pi_{i=1}^n \Lambda_i(x_i)}\Big) \leq (\pi_{i=1}^n k_i)^p \sum_B \pi_{i=1}^n \lambda_i(x_i) H^p(a(x)). \tag{16}$$

Proof. Since H is a convex function, by repeated application of Jensen's inequality, we have

$$H\Big(\frac{A(x)}{\pi_{i=1}^n \Lambda_i(x_i)}\Big) \leq \frac{1}{\pi_{i=1}^n \Lambda_i(x_i)} \sum_{B(e,x)} \pi_{i=1}^n \beta_i(y_i)\lambda_i(y_i) H(a(y))$$

$$= \frac{C(x)}{\pi_{i=1}^n \Lambda_i(x_i)}.$$

where $C(x) = \sum_{B(e,x)} \pi_{i=1}^n \beta_i(y_i)\lambda_i(y_i) H(a(y))$

Thus $\sum_B \pi_{i=1}^n \lambda_i(x_i) H^p\Big(\frac{A(x)}{\pi_{i=1}^n \Lambda_i(x_i)}\Big) \leq \sum_B \pi_{i=1}^n \lambda_i(x_i)\Big(\frac{C(x)}{\pi_{i=1}^n \Lambda_i(x_i)}\Big)^p$.

Replace $a(y)$ by $H(a(y))$ in (9), we have

$$\sum_B \pi_{i=1}^n \lambda_i(x_i) H^p\Big(\frac{A(x)}{\pi_{i=1}^n \Lambda_i(x_i)}\Big) \leq (\pi_{i=1}^n k_i)^p \sum_B \pi_{i=1}^n \lambda_i(x_i) H^p(a(x)).$$

This completes the proof.

Remark 2. The inequality (9) is the special case of the inequality (16) when $H(u) = u$, and Theorem 2 reduces to Theorem E and Theorem G when $n = 1$ and $H(u) = u, K_i = \frac{p}{p-1}, \beta_i(x_i) = 1, \lambda_i(x_i) = 1$, for $i = 1, 2, \ldots, n$. $x \geq 1$, respectively.

Theorem 3. *Let $P, a(x), \beta_i(x_i), \lambda_i(x_i), \Lambda_i(x_i), K_i$, for $i = 1, 2, \ldots, n$ and $A(x)$ be as in Theorem 1 and let $\phi > 0$ be define on $(0, \infty)$ so that $\phi'' > 0$ and*

$$\phi \cdot \phi'' \geq \Big(1 - \frac{1}{p}\Big)(\phi')^2 \tag{17}$$

If $\sum_B \pi_{i=1}^n \lambda_i(x_i)\phi(a(x))$ converges, then

$$\sum_B \pi_{i=1}^n \lambda_i(x_i)\phi\Big(\frac{A(x)}{\pi_{i=1}^n \Lambda_i(x_i)}\Big) \leq (\pi_{i=1}^n k_i)^p \sum_B \pi_{i=1}^n \lambda_i(x_i)\phi(a(x)) \tag{18}$$

The proof of theorem 3 is similar to the proof of theorem F with suitable modification.

Remark 3. Theorem 3 reduces to Theorem 2 and Theorem 1 when $\phi(u) = H^p(u)$ and $\phi(u) = u^p$, respectively. Also we note that the inequality (6) and (7) are the special case of the inequality (18) when $n = 1$ and

$$\phi(u) = u^p, \ K_i = \frac{p}{p-1}, \ \beta_i(x_i) = 1, \ \lambda_i(x_i) = 1,$$

for $i = 1, 2, \ldots, n$, $x_i \geq 1$ respectively.

References

[1] E. T. Copson, "Note on a series of positive terms," *J. London Math. Soc.*, 2(1927), 9-12.
[2] G. H. Hardy, "Note on a Theorem of Hilbert," *Math. Zeitschr*, 6(1920), 314-317.
[3] D. Y. Hwang and G. S. Yang, "Note on discrete Hardy's inequality," *Tamkang J. Math.*, 21(4) (1990), 333-336.
[4] B. G. Pachpatte, "A note on Copson's inequality involving series of positive terms," *Tamkang J. Math.*, 21(1)(1990), 13-19.
[5] B. G. Pachpatte, "A many variable generalization of Hardy's inequality concerning a series of terms," *Tamkang J. Math.*, 23(4)(1992), 349-354.

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