TAMKANG JOURNAL OF MATHEMATICS
Volume 27, Number 2, Summer 1996

ON THE DOUBLE NÖRLUND SUMMABILITY OF DOUBLE FOURIER SERIES

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Abstract. We extend Rajagopal's theorem [12] to a theorem on the double Nörlund summability of double Fourier series, from which various known results are deduced.

1. Let $\{p_n^{(r)}\}(r=1,2)$ be two sequences of constants and let

$$P_n^{(r)}=\sum_{k=0}^{n} p_k^{(r)} \neq 0.$$

The double series $\sum a_{mn}$ with the sequence of partial sum $\{s_{mn}\}$ is said to be summable by double Nörlund method, or summable $(N,p_m^{(1)},p_n^{(2)})$ if t_{mn} tends to a limit as $(m,n)\to\infty$, where the double Nörlund mean t_{mn} is defined by

$$t_{mn}=\frac{1}{P_m^{(1)}P_n^{(2)}}\sum_{i=0}^{m}\sum_{k=0}^{n} p_{m-i}^{(1)}p_{n-k}^{(2)}s_{ik} \tag{1.1}$$

(see Herriot [4]). In the special case in which $p_m^{(1)}=p_n^{(2)}=1$ or $p_m^{(1)}=p_n^{(2)}=\frac{1}{(n+1)}$, the summability $(N,p_m^{(1)},p_n^{(2)})$ is the same as the summability $(C,1,1)$ or the summability $(H,1,1)$, respectively.

Suppose that $f(u,v)$ is integrable (L) over the square $Q(-\pi,\pi;-\pi,\pi)$ and is periodic with period 2π in each variable.

The double Fourier series of function $f(u,v)$ is

$$\begin{aligned}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\lambda_{mn}[a_{mn}\cos mu\cos nv+b_{mn}\sin mu\cos nv\\ +c_{mn}\cos mu\sin nv+d_{mn}\sin mu\sin nv]\\ =\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\lambda_{mn}A_{mn}(u,v),\end{aligned}$$

Received October 29, 1994.
1991 *Mathematics Subject Classification.* Primary 42B05, 42B08.
Key words and phrases. Double Fourier series, double Nörlund summability.

$$\lambda_{00} = \frac{1}{4};$$

$$\left.\begin{array}{l}\lambda_{m0}\\ \lambda_{0n}\\ \lambda_{mn}\end{array}\right\} = \begin{array}{ll}\frac{1}{2} & \text{for } m>0\\ \frac{1}{2} & \text{for } n>0\\ 1 & \text{for } m>0, n>0\end{array}$$

and

$$a_{mn} = \frac{1}{\pi^2}\int\int_Q f(u,v)\cos mu \cos nv du dv$$

and three other similar expressions for b_{mn}, c_{mn} and d_{mn}.

We write

$$\phi(u,v) = \frac{1}{4}[f(x+u,y+v)+f(x+u,y-v)+f(x-u,y+v)+f(x-u,y-v)-4f(x,y)]; \quad (1.2)$$

$$\Phi(u,v) = \int_0^u\int_0^v |\phi(s,t)|dsdt; \quad (1.3)$$

$$\Phi_1(u,t) = \int_0^u |\phi(s,t)|ds; \quad (1.4)$$

$$\Phi_2(s,v) = \int_0^v |\phi(s,t)|dt \quad (1.5)$$

and for $r = 1, 2$

$$K_m^{(r)}(u) = \sum_{k=0}^{m} p_k^{(r)}\frac{\sin(m-k+\frac{1}{2})u}{\sin\frac{1}{2}u}. \quad (1.6)$$

2. Let $f(t)$ be a periodic finite-valued function with 2π and integrable (L) over $(-\pi,\pi)$. We write

$$\phi(t) = \frac{1}{2}\{f(x+t)+f(x-t)-2f(x)\};$$

$$\Phi(t) = \int_0^t |\phi(u)|du.$$

Rajagopal [12] previously proved the following nice theorem on the Nörlund summability of Fourier series.

Theorem A. *Let a function $p(t)$ be monotone nonincreasing and positive for $t \geq 0$. Let $p_n = p(n)$ and let*

$$P(t) \equiv \int_0^t p(u)du \to \infty, \quad \textit{as} \quad t\to\infty.$$

If, for some fixed δ, $0<\delta<1$,

$$\int_{\pi/n}^{\delta} \Phi(t)\left|\frac{d}{dt}\frac{P(\frac{\pi}{t})}{t}\right|dt = o(P_n), \quad \textit{as} \quad n\to\infty,$$

then the Fourier series of function $f(t)$ is summable (N, p_n) to $f(t)$, at the point $t = x$.

Theorem A contains various results due to Hardy [3], Hirokawa [6], Hirokawa and Kayashima [7], Pati [11], Siddiqi [14] and Singh [15,16].

The purpose of this paper is to extend Theorem A to a theorem on the double Nörlund summability of double Fourier series.

Dealing with the harmonic summability of double Fourier series, Sharma [13] proved the following theorem.

Theorem B. *If the conditions*

$$\Phi(u,v) = o(uv/\log 1/u \log 1/v), \tag{2.1}$$

$$\int_0^{\pi} \Phi_1(u,t)dt = O(u/\log 1/u) \tag{2.2}$$

and

$$\int_0^{\pi} \Phi_2(s,v)ds = O(v/\log 1/v) \tag{2.3}$$

hold, then the double Fourier series of function $f(u,v)$ is summable $(H,1,1)$ to $f(u,v)$, at the point $(u,v) = (x,y)$.

This theorem is a generalization of the theorem due to Hille and Tamarkin [5] for double Fourier series and also is analogous to the theorem of Chow [1] for summability $(C,1,1)$ of the double Fourier series.

Generalizing Theorem B, Mishra [10] proved the following theorem.

Theorem C. *Let a function $P^{(r)}(t)(r = 1,2)$ be tending to ∞ with t and a function $p^{(r)}(t)(r = 1,2)$ be monotonic decreasing and strictly positive for $t \geq 0$, such that*

$$P^{(r)}(t) = \int_0^t p^{(r)}(x)dx, \quad p^{(r)}(n) = p_n^{(r)}.$$

If the conditions

$$\Phi(u,v) = o(uv/\Psi^{(1)}(1/u)\Psi^{(2)}(1/v)), \tag{2.4}$$

$$\int_0^{\pi} \Phi_1(u,t)dt = O(u/\Psi^{(1)}(1/u)) \tag{2.5}$$

and

$$\int_0^{\pi} \Phi_2(s,v)ds = O(v/\Psi^{(2)}(1/v)) \tag{2.6}$$

hold, then the double Fourier series of function $f(u,v)$ is summable $(N, p_m^{(1)}, p_n^{(2)})$ to $f(u,v)$, at the point $(u,v) = (x,y)$, where $\Psi^{(r)}(t)(r = 1,2)$ is a positive nondecreasing function with t such that

$$\int_1^n \frac{P^{(r)}(x)}{x\Psi^{(r)}(x)}dx = O(P_n^{(r)}). \tag{2.7}$$

If we put $p_n^{(r)} = 1/(n+1)$ and $\Psi^{(r)}(x) = \log x$ in Theorem C, we can obtain Theorem B from Theorem C.

Though the reviewer ([MR] 87f:42034) pointed out that there appear to be errors in the proof of Theorem C, we think that Theorem C is essentially true.
Now we generalize these Theorems B and C in the following form.

Theorem. *Let a function $P^{(r)}(t)(r = 1,2)$ be tending to ∞ with t and a function $p^{(r)}(t)(r = 1,2)$ be monotonic decreasing and strictly positive for $t \geq 0$, such that*

$$P^{(r)}(t) = \int_0^t p^{(r)}(x)dx, \quad p^{(r)}(n) = p_n^{(r)}.$$

If the conditions

$$\int_{1/m}^{\delta}\int_{1/n}^{\tau} \Phi(u,v)\Big|\frac{d}{du}\frac{P^{(1)}(1/u)}{u}\Big|\Big|\frac{d}{dv}\frac{P^{(2)}(1/v)}{v}\Big|dudv = o(P_m^{(1)}P_n^{(2)}), \quad \text{as } (m,n) \to \infty, \tag{2.8}$$

$$\int_0^{\pi} dt \int_{1/m}^{\delta} \Phi_1(u,t)\Big|\frac{d}{du}\frac{P^{(1)}(1/u)}{u}\Big|du = O(P_m^{(1)}), \quad \text{as } m \to \infty \tag{2.9}$$

and

$$\int_0^{\pi} ds \int_{1/n}^{\tau} \Phi_2(s,v)\Big|\frac{d}{dv}\frac{P^{(2)}(1/v)}{v}\Big|dv = O(P_n^{(2)}), \quad \text{as } n \to \infty \tag{2.10}$$

hold for $0 < \delta, \tau < \pi$, then the double Fourier series of function $f(u,v)$ is summable $(N, p_m^{(1)}, p_n^{(2)})$ to $f(u,v)$, at the point $(u,v) = (x,y)$.

If the condition (2.4) holds, then we have by (2.7)

$$\begin{aligned}
&\int_{1/m}^{\delta}\int_{1/n}^{\tau} \Phi(u,v)\Big|\frac{d}{du}\frac{P^{(1)}(1/u)}{u}\Big|\Big|\frac{d}{dv}\frac{P^{(2)}(1/v)}{v}\Big|dudv \\
&= o\Big(\int_{1/m}^{\delta}\int_{1/n}^{\tau} \frac{uv}{\Psi^{(1)}(1/u)\Psi^{(2)}(1/v)} \cdot \frac{P^{(1)}(1/u)}{u^2} \cdot \frac{P^2(1/v)}{v^2} dudv\Big) \\
&= o\Big(\int_{1/m}^{\delta} \frac{P^{(1)}(1/u)}{u\Psi^{(1)}(1/u)}du \int_{1/n}^{\tau} \frac{P^{(2)}(1/v)}{v\Psi^{(2)}(1/v)}dv\Big) \\
&= o\Big(\int_{1/\delta}^{m} \frac{P^{(1)}(x)}{x\Psi^{(1)}(x)}dx \int_{1/\tau}^{n} \frac{P^{(2)}(y)}{y\Psi^{(2)}(y)}dy\Big) \\
&= o\Big(P_m^{(1)}P_n^{(2)}\Big)
\end{aligned}$$

by virture of the fact that $\frac{d}{du}\Big(\frac{P^{(r)}(1/u)}{u}\Big) = O\Big(\frac{P^{(r)}(1/u)}{u^2}\Big)$.

Similarly, we have by the conditions (2.5) and (2.7)

$$\begin{aligned}
&\int_0^\pi dt \int_{1/m}^{\delta} \Phi_1(u,t)\Big|\frac{d}{du}\Big(\frac{P^{(1)}(1/u)}{u}\Big)\Big|du \\
&= \int_{1/m}^{\delta}\int_0^\pi \Phi_1(u,t)dt\Big|\frac{d}{du}\frac{P^{(1)}(1/u)}{u}\Big|du \\
&= O\Big(\int_{1/m}^{\delta} \frac{u}{\Psi^{(1)}(1/u)}\cdot\frac{P^{(1)}(1/u)}{u^2}du\Big) \\
&= O\Big(\int_{1/\delta}^{m} \frac{P^{(1)}(x)}{x\Psi^{(1)}(x)}dx\Big) \\
&= O\big(P_m^{(1)}\big).
\end{aligned}$$

Also, the conditions (2.6) and (2.7) imply the condition (2.10). Thus we see that our theorem is a generalization of Theorems B and C.

3. We need some lemmas for the proof of our Theorem.

Lemma 1 [9]. *If a sequence $\{p_n^{(r)}\}(r=1,2)$ is nonnegative and nonincreasing, then we have*

$$\Big|\sum_{k=0}^{n} p_k^{(r)}\sin(n-k+\frac{1}{2})u\Big| \le CP^{(r)}(1/u),$$

where C is a positive constant.

Lemma 2. *(i) The condition (2.8) implies the conditon $\Phi(u,v)=o(uv)$. (ii) The condition (2.9) or (2.10) implies the condition $\int_o^\pi \Phi_1(1/m,t)dt = O(1/m)$ or $\int_o^\pi \Phi_2(s,1/n)ds = O(1/n)$, respectively.*

Proof. (i) By the condition (2.8), we have

$$\begin{aligned}
o(P_m^{(1)}P_n^{(2)}) &= \int_{1/m}^{\delta}\int_{1/n}^{\tau} \Phi(u,v)\Big|\frac{d}{du}\frac{P^{(1)}(1/u)}{u}\Big|\Big|\frac{d}{dv}\frac{P^{(2)}(1/v)}{v}\Big|dudv \\
&\ge \Phi(1/m,1/n)\int_{1/m}^{\delta}\Big|\frac{d}{du}\frac{P(1/u)}{u}\Big|du\int_{1/n}^{\pi}\Big|\frac{d}{dv}\frac{P^{(2)}(1/v)}{v}\Big|dv \\
&= \Phi(1/m,1/n)\Big\{-\int_{1/m}^{\delta}\frac{d}{du}\frac{P^{(1)}(1/u)}{u}du\Big\}\Big\{-\int_{1/n}^{\tau}\frac{d}{dv}\frac{P^{(2)}(1/v)}{v}dv\Big\} \\
&= \Phi(1/m,1/n)\Big\{mP^{(1)}(m)-\frac{1}{\delta}P^{(1)}(1/\delta)\Big\}\Big\{nP^{(2)}(n)-\frac{1}{\tau}P^{(2)}(1/\tau)\Big\} \\
&\sim mn\Phi(1/m,1/n)P^{(1)}(m)P^{(2)}(n).
\end{aligned}$$

Hence we have $\Phi(1/m,1/n)=o(1/mn)$. Since $\Phi(u_1,v_1)\le\Phi(u_2,v_2)$ for $u_1\le u_2$ and $v_1\le v_2$, we obtain $\Phi(u,v)=o(uv)$.

(ii). By the condition (2.10), we have

$$
\begin{aligned}
O(P_n^{(2)}) &= \int_0^\pi ds \int_{1/n}^\tau \Phi_2(s,v)\Big|\frac{d}{dv}\frac{P^{(2)}(1/v)}{v}\Big|dv \\
&= \int_{1/n}^\tau \Big|\frac{d}{dv}\frac{P^{(2)}(1/v)}{v}\Big|dv \int_0^\pi \Phi_2(s,v)ds \\
&= \int_{1/n}^\tau \Big|\frac{d}{dv}\frac{P^{(2)}(1/r)}{v}\Big|dv \int_0^\pi ds \int_0^v |\phi(s,t)|dt \\
&\geq \int_{1/n}^\tau \Big|\frac{d}{dv}\frac{P^{(2)}(1/v)}{v}\Big|dv \int_0^\pi ds \int_0^{1/n} |\phi(x,t)|dt \\
&= -\int_{1/n}^\tau \frac{d}{dv}\frac{P^{(2)}(1/v)}{v}dv \int_0^\pi \Phi_2(s,1/n)ds \\
&= \left(nP^{(2)}(n) - \frac{1}{\tau}P^{(2)}(1/\tau)\right)\int_0^\pi \Phi_2(s,1/n)ds \\
&\sim nP^{(2)}(n)\int_0^\pi \Phi_2(s,1/n)ds.
\end{aligned}
$$

Thus we have $\int_0^\pi \Phi_2(s,1/n)ds = O(1/n)$. The other case is similarly proved.

4. Proof of Theorem. By (1.1), we have

$$
\begin{aligned}
\pi^2 t_{mn} &= \frac{1}{P_m^{(1)}P_n^{(2)}}\int_0^\pi\int_0^\pi \phi(s,t)K_m^{(1)}(s)K_n^{(2)}(t)dsdt \\
&= \frac{1}{P_m^{(1)}P_n^{(2)}}\Big[\int_0^\delta\int_0^\tau + \int_0^\delta\int_\tau^\pi + \int_\delta^P\int_0^\tau + \int_\delta^\pi\int_\tau^\pi\Big]\phi(s,t)K_m^{(1)}(s)K_n^{(2)}(t)dsdt \\
&\equiv I_1 + I_2 + I_3 + I_4,
\end{aligned}
$$

say.

Now let $m^{-1} < \delta < \pi, n^{-1} < \tau < \pi$. Then we obtain

$$
\begin{aligned}
I_3 \leq & \frac{1}{P_m^{(1)}P_n^{(2)}}\int_\delta^\pi\int_0^\tau |\phi(s,t)||K_m^{(1)}(s)||K_n^{(2)}(t)|dsdt \\
= & \frac{1}{P_m^{(1)}P_n^{(2)}}\int_\delta^\pi |K_m^{(1)}(s)|ds \int_0^{1/n} |\phi(s,t)||K_n^{(2)}(t)|dt \\
& + \frac{1}{P_m^{(1)}P_n^{(2)}}\int_\delta^\pi |K_m^{(1)}(s)|ds \int_{1/n}^\tau |\phi(s,t)||K_n^{(2)}(t)|dt \\
= & I_{31} + I_{32},
\end{aligned}
$$

say. By Lemmas 1 and 2, we have

$$\begin{aligned}
I_{31} &= O\Big(\frac{1}{P_m^{(1)}P_n^{(2)}}\int_\delta^\pi \frac{P^{(1)}(1/s)}{s}ds\int_0^{1/n}|\phi(s,t)|O(nP_n^{(2)})dt\Big)\\
&= O\Big(\frac{n}{P_m^{(1)}}\int_0^\pi \Phi_2(s,1/n)ds\Big)ds\\
&= O\Big(\frac{1}{P_m^{(1)}}\Big)\\
&= o(1), \quad \text{as } (m,n)\to\infty.
\end{aligned}$$

Applying Lemma 1 and integrating by parts, we obtain

$$\begin{aligned}
I_{32} &\le \frac{1}{P_m^{(1)}P_n^{(2)}}\int_\delta^\pi |K_m^{(1)}(s)ds|\int_{1/n}^\tau |\phi(s,t)|\,|K_n^{(2)}(t)|dt\\
&= O\Big[\frac{1}{P_m^{(1)}P_n^{(2)}}\int_\delta^\pi \frac{P^{(1)}(1/s)}{s}ds\int_{1/n}^\tau |\phi(s,t)|\frac{P^{(2)}(1/t)}{t}dt\Big]\\
&= O\Big[\frac{1}{P_m^{(1)}P_n^{(2)}}\int_\delta^\pi ds\Big\{\Big[\Phi_2(s,t)\frac{P^{(2)}(1/t)}{t}\Big]_{1/n}^\tau - \int_{1/n}^\tau \Phi_2(s,t)\frac{d}{dt}\frac{P^{(2)}(1/t)}{t}dt\Big\}\Big]\\
&= O\Big(\frac{1}{P_m^{(1)}P_n^{(2)}}\frac{P^{(2)}(1/\tau)}{\tau}\int_\delta^\pi \Phi_2(s,\tau)ds\Big) + O\Big(\frac{n}{P_m^{(1)}}\int_\delta^\pi \Phi_2(s,1/n)ds\Big)\\
&\quad + O\Big(\frac{1}{P_m^{(1)}P_n^{(2)}}\int_\delta^\pi ds\int_{1/n}^\tau \Phi_2(s,t)\Big|\frac{d}{dt}\frac{P^{(2)}(1/t)}{t}\Big|dt\Big)\\
&= O(I_{321}+I_{322}+I_{323}),
\end{aligned}$$

say. Clearly we get $I_{321}=o(1)$. By Lemma 2, we have

$$\begin{aligned}
I_{322} &= \frac{n}{P_m^{(1)}}\int_\delta^\pi \Phi_2(s,1/n)ds\\
&= O\Big(\frac{1}{P_m^{(1)}}\Big)\\
&= o(1), \quad \text{as } (m,n)\to\infty.
\end{aligned}$$

By the condition (2.10), we have

$$\begin{aligned}
I_{323} &= O\Big(\frac{P_n^{(2)}}{P_m^{(1)}P_n^{(2)}}\Big)\\
&= O\Big(\frac{1}{P_m^{(1)}}\Big)\\
&= o(1), \quad \text{as } (m,n)\to\infty.
\end{aligned}$$

Thus we get $I_3=o(1)$. Similarly, we get $I_2=o(1)$.

Moreover, since $K_m^{(1)}(s)$ and $K_n^{(2)}(t)$ are bounded on $[\delta, \pi]$ and $[\tau, \pi]$ respectively, we have

$$
\begin{aligned}
I_4 &= O\Big(\frac{1}{P_m^{(1)} P_n^{(2)}} \int_\delta^\pi \int_\tau^\pi |\phi(s,t)||K_m^{(1)}(s)||K_n^{(2)}(t) ds dt\Big) \\
&= O\Big(\frac{1}{P_m^{(1)} P_n^{(2)}} \int_\delta^\pi \int_\tau^\pi |\phi(s,t)| ds dt\Big) \\
&= o(1), \quad \text{as } (m,n) \to \infty.
\end{aligned}
$$

Finally we obtain

$$
\begin{aligned}
I_1 \le & \frac{1}{P_m^{(1)} P_n^{(2)}} \Big[\int_0^{1/m} \int_0^{1/n} + \int_{1/m}^{\delta} \int_0^{1/n} + \int_0^{1/m} \int_{1/n}^{\tau} \\
& + \int_{1/m}^{\delta} \int_{1/n}^{\tau} \Big] |\phi(s,t)||K_m^{(1)}(t)||K_n^{(2)}(s)| ds dt \\
= & I_{11} + I_{12} + I_{13} + I_{14},
\end{aligned}
$$

say. By Lemma 2, we have

$$
\begin{aligned}
I_{11} \le & \frac{1}{P_m^{(1)} P_n^{(2)}} \int_0^{1/m} \int_0^{1/n} |\phi(s,t)| O(mn P_m^{(1)} P_n^{(2)}) ds dt \\
= & O(mn \int_0^{1/m} \int_0^{1/n} |\phi(s,t)| ds dt) \\
= & o(1), \quad \text{as } (m,n) \to \infty.
\end{aligned}
$$

Appling Lemma 1 and integrating by parts, we obtain

$$
\begin{aligned}
I_{12} &\le \frac{1}{P_m^{(1)} P_n^{(2)}} \int_0^{1/n} |K_n^{(2)}(t)| dt \int_{1/m}^{\delta} |\phi(s,t)||K_m^{(1)}(s)| ds \\
&= O\Big(\frac{1}{P_m^{(1)} P_n^{(2)}} \int_0^{1/n} (n P_n^{(2)}) dt \int_{1/m}^{\delta} |\phi(s,t)| \frac{P^{(1)}(1/s)}{s} ds\Big) \\
&= O\Big(\frac{n}{P_m^{(1)}} \int_0^{1/n} dt \int_{1/m}^{\delta} |\phi(s,t)| \frac{P^{(1)}(1/s)}{s} ds\Big) \\
&= O\Big(\frac{n}{P_m^{(1)}} \int_0^{1/n} dt \Big\{ \Big[\Phi_1(s,t) \frac{P^{(1)}(1/s)}{s} \Big]_{1/m}^{\delta} - \int_{1/m}^{\delta} \Phi_1(s,t) \frac{d}{ds}\Big(\frac{P^{(1)}(1/s)}{s}\Big) ds \Big\}\Big) \\
&= O\Big(\frac{n}{P_m^{(1)}} \int_0^{1/n} \Phi_1(\delta,t) \frac{P^{(1)}(1/\delta)}{\delta} dt + \frac{n}{P_m^{(1)}} \frac{P^{(1)}(m)}{\frac{1}{m}} \int_0^{1/n} \Phi_1(1/m,t) dt \\
&\quad + \frac{n}{P_m^{(1)}} \int_0^{1/n} dt \int_{1/m}^{\delta} \Phi_1(s,t) \Big| \frac{d}{ds} \frac{P^{(1)}(1/s)}{s} \Big| ds\Big) \\
&= O(I_{121} + I_{122} + I_{123}),
\end{aligned}
$$

say. By Lemma 2, we have

$$\begin{aligned} I_{121} &= O\Big(\frac{n}{P_m}\int_0^{1/n}\Phi_1(\delta,t)dt\Big) \\ &= O\Big(\frac{n}{P_m}\Phi(\delta,1/n)\Big) \\ &= O\Big(\frac{n}{P_m}o(\frac{1}{n})\Big) \\ &= o\Big(\frac{1}{P_m}\Big) \\ &= o(1), \quad \text{as } (m,n)\to\infty \end{aligned}$$

and

$$\begin{aligned} I_{122} &= O\Big(mn\int_0^{1/n}\Phi_1(1/m,t)dt\Big) \\ &= O(mn\Phi(1/m,1/n)) \\ &= o(1), \quad \text{as } (m,n)\to\infty. \end{aligned}$$

On the other hand, we have by the conditoin (2.8)

$$\begin{aligned} o(1) &= \frac{1}{P_m^{(1)}P_n^{(2)}}\int_{1/m}^{\delta}\int_{1/n}^{\tau}\Phi(u,v)\Big|\frac{d}{du}\frac{P^{(1)}(1/u)}{u}\Big|\Big|\frac{d}{dv}\frac{P^{(2)}(1/v)}{v}\Big|dudv \\ &\geq \frac{1}{P_m^{(1)}P_n^{(2)}}\int_{1/m}^{\delta}\Phi(u,1/n)\Big|\frac{d}{du}\frac{P^{(1)}(1/u)}{u}\Big|du\int_{1/n}^{\tau}\Big|\frac{d}{du}\frac{P^{(2)}(1/v)}{v}\Big|dv \\ &\geq A\frac{nP_n^{(2)}}{P_m^{(1)}P_n^{(2)}}\int_{1/m}^{\delta}\Phi(u,1/n)\Big|\frac{d}{du}\frac{P^{(1)}(1/u)}{u}\Big|du \\ &\geq A\frac{n}{P_m^{(1)}}\int_0^{1/n}dt\int_{1/m}^{\delta}\Phi_1(u,t)\Big|\frac{d}{du}\frac{P^{(1)}(1/u)}{u}\Big|du \\ &= AI_{123}. \end{aligned}$$

Thus we get $I_{123} = o(1)$. Hence we have $I_{12} = o(1)$. Similarly, we have $I_{13} = o(1)$.

By partial integration for double integral [2,8] and Lemma 1, we have

$$I_{14} \leq \frac{1}{P_m^{(1)}P_n^{(2)}}\int_{1/m}^{\delta}\int_{1/n}^{\tau}|\phi(s,t)|\frac{P^{(1)}(1/s)}{s}\frac{P^{(2)}(1/t)}{t}dtds$$

$$
\begin{aligned}
&= \frac{1}{P_m^{(1)}P_n^{(2)}}\Phi(\delta,\tau)\frac{P^{(1)}(1/\delta)}{\delta}\frac{P^{(2)}(1/\tau)}{\tau}\\
&- \frac{1}{P_m^{(1)}P_n^{(2)}}\frac{P^{(2)}(1/\tau)}{\tau}\int_{1/m}^{\delta}\Phi(u,\tau)\frac{d}{du}\Big(\frac{P^{(1)}(1/u)}{u}\Big)du\\
&- \frac{1}{P_m^{(1)}P_n^{(2)}}\frac{P^{(1)}(1/\delta)}{\delta}\int_{1/n}^{\delta}\Phi(\delta,v)\frac{d}{dv}\Big(\frac{P^{(2)}(1/v)}{v}\Big)dv\\
&+ \frac{1}{P_m^{(1)}P_n^{(2)}}\int_{1/m}^{\delta}\int_{1/n}^{\tau}\Phi(u,v)\frac{d}{du}\Big(\frac{P^{(1)}(1/u)}{u}\Big)\frac{d}{dv}\Big(\frac{P^{(2)}(1/v)}{v}\Big)dudv\\
&=I_{141}+I_{142}+I_{143}+I_{144},
\end{aligned}
$$

say. Clearly we get $I_{141}=o(1)$. By the condition (2.8), we have $I_{144}=o(1)$. Also, by the condition (2.9), we have

$$
\begin{aligned}
I_{142} &\le \frac{1}{P_m^{(1)}P_n^{(2)}}\frac{P^{(2)}(1/\tau)}{\tau}\int_{1/m}^{\delta}\Phi(u,\tau)\Big|\frac{d}{du}\frac{P^{(1)}(1/u)}{u}\Big|du\\
&= O\Big(\frac{1}{P_m^{(1)}P_n^{(2)}}\int_0^{\tau}dt\int_{1/m}^{\delta}\Phi_1(u,t)\Big|\frac{d}{du}\frac{P^{(1)}(1/u)}{u}\Big|du\Big)\\
&= O\Big(\frac{1}{P_m^{(1)}P_n^{(2)}}\int_0^{\pi}dt\int_{1/m}^{\delta}\Phi_1(u,t)\Big|\frac{d}{du}\frac{P^{(1)}(1/u)}{u}\Big|du\Big)\\
&= O\Big(\frac{1}{P_m^{(1)}P_n^{(2)}}O(P_m^{(1)})\Big)\\
&= O\Big(\frac{1}{P_n^{(2)}}\Big)\\
&= o(1), \quad \text{as } (m,n)\to\infty.
\end{aligned}
$$

Similarly, we get $I_{143}=o(1)$. Hence we have $I_{14}=o(1)$. Therefore, by the above estimations, our theorem is completely proved.

5. In this section, we deduce some corollaries from our theorem.

Corollary 1. *If the condtions*

$$\Phi(u,v)=o(uv),$$

$$\Phi(u,\pi)=O(u)$$

and

$$\Phi(\pi,v)=O(v)$$

hold, then the double Fourier series of function $f(u,v)$ is summable $(C,1,1)$ to $f(u,v)$, at the point $(u,v)=(x,y)$.

A weaker form of Corollary 1 was proved by Chow [1].

Corollary 2. *Let $p^{(r)}(t)$ and $P^{(r)}(t)(\tau = 1, 2)$ be the functions satisfying the same hypothesis of Theorem such that*

$$\log n = O(P_n^{(r)})(r = 1, 2).$$

If the conditions

$$\Phi(u, v) = o(uv/P^{(1)}(1/u)P^{(2)}(1/v)),$$

$$\Phi(u, \pi) = O(u/P^{(1)}(1/u))$$

and

$$\Phi(\pi, v) = O(v/P^{(2)}(1/v))$$

hold, then the double Fourier series of function (u, v) is summable $(N, p_m^{(1)}, p_n^{(2)})$ to $f(u, v)$, at the point $(u, v) = (x, y)$.

This corollary is also deduced from Theorem C.

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