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# TOTALLY UMBILICAL CR-SUBMANIFOLDS OF A NEARLY KAEHLER MANIFOLD

S. H. KON AND SIN-LENG TAN

**Abstract.** The geometry of a CR-submanifold in a Kaehler manifold has been extensively studied. B.Y. Chen has classified the totally umbilical CR-submanifolds of a Kaehler manifold and showed that they are either totally geodesic, or totally real or $\dim(D^{\perp}) = 1$. In this paper we show that such a result is also true in a nearly Kaehler manifold.

## 1. Introduction

A. Bejancu [1] introduced the notion of a CR-submanifold of an almost Hermitian manifold. The geometry of a CR-submanifold in a Kaehler manifold has been extensively studied, a number of these results also hold for a CR-submanifold of a nearly Kaehler manifold, see [2], [4], [6] and [7].

B. Y. Chen [4] classified the totally umbilical CR-submanifolds of a Kaehler manifold and showed that they are either totally geodesic, or totally real or $\dim(D^{\perp}) = 1$. In this paper we shall generalize this result to nearly Kaehler manifolds and also show that the anti-holomorphic distribution of a totally umbilical CR-submanifold in a nearly Kaehler manifold is integrable and its leaves are totally geodesic.

## 2. Preliminaries

Let $N$ be an almost Hermitian manifold with almost complex structure $J$ and Hermitian metric $g$. A submanifold $M$ of $N$ is said to be a CR-submanifold of $N$ if there exists a differentiable distribution.

$$D : x \to D_x \subset T_xM$$

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on $M$ satisfying the following conditions:
(i) $D$ is holomorphic, i.e., $J(D_x) = D_x$ for each $x \in M$.
(ii) the complementary orthogonal distribution

$$D^{\perp} : x \to D_x^{\perp} \subset T_xM.$$

is anti-invariant, i.e., $J(D_x^{\perp}) \subset T_xM^{\perp}$ for each $x \in M$.

If $D = \{0\}$, (resp. $D^{\perp} = \{0\}$), then $M$ is said to be a totally real (resp. holomorphic) submanifold. The normal bundle $TM^{\perp}$ splits as $TM^{\perp} = JD^{\perp} \oplus \mu$, where $\mu$ is the orthogonal complement of $JD^{\perp}$ and is an invariant subbundle of $TM^{\perp}$ under $J$.

Let $\tilde{\nabla}$ be the Riemannian connection on $N$, then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$
$$\tilde{\nabla}_X U = -A_U X + \nabla_X^{\perp} U.$$

for $X, Y \in \Gamma(TM)$ and $U \in \Gamma(TM^{\perp})$, where $\nabla$ is the induced Riemannian connection on $M$,$h$ the second fundamental form of $M$, $A_U$ the second fundamental tensor related by

$$g(A_U X, Y) = g(h(X, Y), U)$$

and $\nabla^{\perp}$ is the induced normal connection in the normal bundle $TM^{\perp}$.

A submanifold $M$ is said to be totally umbilical if

$$h(X, Y) = g(X, Y)H \quad \text{for all} \quad X, Y \in \Gamma(TM),$$

where $H = \frac{1}{n}$ (trace of $h$), called the mean curvature vector of $M$ in $N$. The submanifold $M$ is a totally geodesic submanifold of $N$ if $h(X, Y) = 0$ for all $X, Y \in \Gamma(TM)$. It is a minimal submanifold if $H = 0$. Obviously a minimal, totally umbilical submanifold is totally geodesic.

A Hermitian manifold $N$ is called a Kaehler manifold if its fundamental 2-form $\Omega$, where $\Omega(X, Y) = g(X, JY)$ for $X, Y \in \Gamma(TN)$, is closed. It is not difficult to show that an almost Hermitian manifold $N$ is a Kaehler manifold if and only if the almost complex structure $J$ is parallel with respect to $\tilde{\nabla}$, i.e., $\tilde{\nabla}_X J = 0$ for all $X \in \Gamma(TN)$. An almost Hermitian manifold is called a nearly Kaehler manifold if we have

$$(\tilde{\nabla}_X J)X = 0 \quad \text{for all} \quad X \in \Gamma(TN).$$

Thus, an almost Hermitian manifold $N$ is nearly Kaehler if and only if

$$(\tilde{\nabla}_X J)Y + (\tilde{\nabla}_Y J)X = 0 \quad \text{for all} \quad X, Y \in \Gamma(TN).$$

## 3. Integrability of the anti-holomorphic distribution of totally umbilical CR-submanifolds

We will prove in this section an integrability theorem on the anti-holomorphic distribution $D^{\perp}$. We first recall two results on the distribution $D^{\perp}$, see [2, page 28].

**Lemma 3.1.** (Sato). *Let $M$ be a CR-submanifold of a nearly Kaehler manifold $N$. The distribution $D^{\perp}$ is integrable if and only if*

$$g(h(U,X),JW) = g(h(W,X),JU) \tag{3.1}$$

*for all $U, W \in \Gamma(D^{\perp})$ and $X \in \Gamma(D)$.*

**Lemma 3.2** (Bejancu). *Let $M$ be a CR-submanifold of a nearly Kaehler manifold $N$. If $D^{\perp}$ is integrable, then each leaf of $D^{\perp}$ is immersed in $M$ as a totally geodesic submanifold if and only if $g(h(U,X),JW) = 0$ for all $U, W \in \Gamma(D^{\perp})$ and $X \in \Gamma(D)$.*

From the above two lemmas, we are able to obtain the following proposition, which-generalizes [3, Lemma 8.2].

**Proposition 3.3.** Let $M$ be a totally umbilical CR-submanifold of a nearly Kaehler manifold $N$. Then $D^{\perp}$ is integrable and its leaves are totally geodesic in $M$.

**Proof.** Since $M$ is totally umbilical in $N$, both sides of (3.1) vanish. It follows from the above two lemmas that $D^{\perp}$ is integrable and its leaves are totally geodesic in $M$.

## 4. The geometry of totally umbilical CR-submanifolds

In this section, we will generalize a classification theorem of Chen [4] to nearly Kaehler manifolds. The following generalizes [3; Lemma 7.1].

**Proposition 4.1.** Let $M$ be a totally umbilical CR-submanifold of a nearly Kaehler manifold $N$. If $\dim D^{\perp} > 1$, then we have
(i) $H \perp JD^{\perp}$,
(ii) $A_{JX}Y = 0$ for all $X, Y \in \Gamma(D^{\perp})$.

**Proof.** (i) Since $M$ is nearly Kaehler, for $Z, W \in \Gamma(D^{\perp})$, we have $(\tilde{\nabla}_Z J)W = -(\tilde{\nabla}_W J)Z$ or $\tilde{\nabla}_Z JW - J\tilde{\nabla}_Z W = -\tilde{\nabla}_W JZ + J\tilde{\nabla}_W Z$
or $-A_{JW}Z + \nabla^{\perp}_Z JW - J\nabla_Z W - Jh(Z,W) = A_{JZ}W - \nabla^{\perp}_W JZ + J\nabla_W Z + Jh(Z,W)$.
Since $M$ is totally umbilical,we have

$$J(\nabla_W Z + \nabla_Z W) + 2g(Z,W)JH = -A_{JW}Z - A_{JZ}W + \nabla^{\perp}_Z JW + \nabla^{\perp}_W JZ.$$

Hence

$$\begin{aligned}&g(J(\nabla_W Z+\nabla_Z W),Z)+2g(Z,W)g(Z,JH)\\&=-\,g(A_{JW}Z,Z)-g(A_{JZ}W,Z)+g(\nabla^{\perp}JW,Z)+g(\nabla^{\perp}_{W}JZ,Z).\end{aligned}$$

Therefore,

$$\begin{aligned}&2g(Z,W)g(Z,JH)\\&=-\,g(h(Z,Z),JW)-g(h(W,Z),JZ)\\&=\|Z\|^2g(W,JH)+g(Z,W)g(Z,JH),\end{aligned}$$

or

$$g(Z,W)g(Z,JH)=\|Z\|^2g(W,JH).$$

Interchanging $Z, W$ in the above equation , we obtain

$$g(Z,W)g(W,JH)=\|W\|^2g(Z,JH).$$

Hence

$$g(W,JH)=\frac{g(Z,W)^2}{\|Z\|^2\|W\|^2}g(W,JH).$$

Now, if $\dim D^{\perp}>1$, then for $Z$ not parallel with $W$,

$$g(Z,W)^2<\|Z\|^2\|W\|^2$$

and so $g(W,JH)=0$, hence $H\perp JD^{\perp}$. (ii) Let $X,Y\in\Gamma(D^{\perp})$ and $Z\in\Gamma(TM)$. Then

$$\begin{aligned}g(A_{JX}Y,Z)&=g(h(Y,Z),JX)=g(Y,Z)g(H,JX)\\&=0\quad\text{from (i)}.\end{aligned}$$

**Remark.** A totally umbilical anti-holomorphic submanifold of a Kaehler manifold is either totally geodesic or a hypersurface [3, Theorem 7.2]. Similarly , if $M$ is a submanifold of a nearly Kaehler manifold $N$ such that $J(T_xM^{\perp})\subset T_xM$ for all $x\in M$, then $M$ can be regarded as a CR-submanifold with $D^{\perp}=J(T_xM^{\perp})$. For such a manifold, which is also totally umbilical, Proposition 4.1 show that either $M$ is totally geodesic or $M$ is a hypersurface in $N$, generalizing the above mentioned result of Blair and Chen for a totally umbilical anti-holomorphic submanifold of a Kaehler manifold.

The following theorem generalizes Chen's classification theorem [4] to nearly Kaehler manifolds.

**Theorem 4.2** *Let $M$ be a totally umbilical CR-submanifold of a nearly Kaehler manifold $N$. Then either*
(i) *$M$ is totally geodesic, or*
(ii) *$M$ is totally real, or*
(iii) *$D^{\perp}$ is of dimension 1.*

**Proof.** If $\dim D^\perp > 1$, we have $JH \in \Gamma(\mu)$ from Proposition 4.1. Now if $M$ is not totally real, then $\dim D \geq 2$. Consider $Z \neq 0$ in $\Gamma(D)$, we have

$$\begin{aligned}
0 &= g(JZ, Z)g(JH, H) \\
&= g(A_{JH}JZ, Z) \\
&= g(-\tilde{\nabla}_{JZ}JH + \nabla^\perp_{JZ}JH, Z) \\
&= g(-\tilde{\nabla}_{JZ}JH, Z) = g(JH, \tilde{\nabla}_{JZ}Z), \quad \text{since} \quad JH \perp Z \\
&= g(JH, -\tilde{\nabla}_{JZ}J(JZ)) \\
&= g(JH, -(\tilde{\nabla}_{JZ}J)JZ - J\tilde{\nabla}_{JZ}JZ) \\
&= g(JH, -J\tilde{\nabla}_{JZ}JZ), \quad \text{since} \quad (\tilde{\nabla}_{JZ}J)JZ = 0 \\
&= g(JH, -J\nabla_{JZ}JZ - Jh(JZ, JZ)) \\
&= g(JH, -Jh(JZ, JZ)) \\
&= -g(JH, JH)g(JZ, JZ) = -\|H\|^2\|Z\|^2
\end{aligned}$$

Hence $H = 0$. Thus $M$ is minimal and hence totally geodesic in $N$. Finally, for the case $D^\perp = \{0\}$, $M$ is mininal, see [7], and so is totally geodesic.

**Remark.** A CR-submanifold $M$ with $D^\perp$ integrable is said to be $D^\perp$-totally umbilical if

$$h(U, W) = g(U, W)H_{D^\perp}$$

for all $U, W \in \Gamma(D^\perp)$ and some vector field $H_{D^\perp}$. A classification theorem, simsilar to Chen's result [4] has been obtained in [5] for $D^\perp$-totally umbilical CR-submanifold of a Kaehler manifold. It is easy to show that when $D^\perp$ is integrable, such a result is also true in a nearly Kaehler manifold.

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Department of Mathematics, University of Malaya, 50603 Kuala Lumpur, Malaysia.