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ON THE OSCILLATION OF AN ELLIPTIC EQUATION OF FOURTH ORDER

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Abstract. The elliptic equation

$$\Delta^2 u(|x|) + g(|x|)u(|x|) = f(|x|)$$

is studied for its oscillatory behavior. Δ is the Laplace operator. Sufficient conditions have been found to ensure that all solutions of this equation continuable in some exterior domain

$$\Omega = \{x = (x_1, x_2, x_3) : |x| > A\}$$

Where

$$|x| = (\sum_{i=1}^{3} x_i^2)^{\frac{1}{2}}.$$

are oscillatory.

I. Introduction

Our main purpose in this work is to study the elliptic equation

$$\Delta^2 u(|x|) + g(|x|)u(|x|) = f(|x|) \tag{1}$$

for its oscillatory behavior in a domain Ω of R^3 external to the hypersphere

$$\sum_{i=1}^{3} x_i^2 = A^2 \tag{2}$$

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where $A > 0$ is sufficiently large and $|x|$ is the Euclidian length

$$|x| = \sqrt{\sum_{i=1}^{3} x_i^2}, \quad (3)$$

$g(t)$ and $f(t)$ are continuous functions on $[A, \infty)$ for some $A > 0$. In equation (1), Δ is the Laplace operator so that

$$\Delta u(x_1, x_2, x_3) = \sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2}. \quad (4)$$

By a solution of (1) we mean a symmetric function $u(|x|)$ which is continuous in some exterior domain

$$\Omega_T = \{x = (x_1, x_2, x_3) \in R^3 : |x| \geq T\}, T \geq A.$$

and satisfies (1).

A function $S(t)$ continuous on $[T, \infty)$ is said to be oscillatory (as in [5], [6]) if $S(t)$ has arbitrarily large zeros in $[T, \infty)$; otherwise $S(t)$ is said to be nonoscillatory. $S(t)$ is said to be slowly oscillating if the set

$$\{|t_m - t_n| : S(t_m) = S(t_n) = 0, S(t) \neq 0 \quad \text{for} \quad t \in (t_m, t_n)\}$$

is unbounded.

We will soon see that equation (1) is closely related to the general ordinary differential equation

$$L_n y(t) + F(t, y(t)) = f(t) \quad (5)$$

which has been, in less general cases, studied in [1-3] and [5-6]. Our techniques and notations in [4-5] will be adapted in this work. Throughout this work we assume that

(i) $L_0 y(t) = \frac{y(t)}{p_0(t)}, L_i y(t) = \frac{1}{p_i(t)} \frac{d}{dt} L_{i-1} y(t)$ for $1 \leq i \leq n$; $\quad n \geq 2$;

(ii) $p_i(t), f(t)$ and $g(t)$ are continuous, real valued on $[A, \infty)$, $1 \leq i \leq n$, $p_i > 0$, $i = 0, 1, 2, \cdots, n$; and

$$\int^{\infty} p_i(t) dt = \infty \quad \text{for} \quad 1 \leq i \leq n-1; \quad (6)$$

(iii) $F : R \times R \to R$ is continuous.

We define as in [5]

$$\left[\begin{array}{c} I_0 = 1 \\ I_k(t, s; p_k, \cdots, p_1) = \int_s^t p_k(r) I_{k-1}(r, s; p_{k-1}, \cdots, p_1) dr. \end{array} \right. \quad (7)$$

It can be easily verified that

$$I_k(t, s; p_k, \cdots, p_1) = \int_s^t p_1(r) I_{k-1}(t, r; p_k, \cdots, p_2) dr, \tag{8}$$

and

$$I_k(t, s; p_k, \cdots, p_1) = (-1)^k I_k(s, t; p_1, \cdots, p_k), \tag{9}$$

$$\left[\begin{array}{l} J_k(t, s) = p_0(t) I_k(t, s; p_1, \cdots, p_k), \\ K_k(t, s) = p_n(t) I_k(t, s; p_{n-1}, \cdots, p_{n-k}). \end{array}\right. \tag{10}$$

$$J_k(t) = J_k(t, A), \quad K_k(t) = K_k(t, A), \quad 0 \le K \le n. \tag{11}$$

It is easy to see that if $p_i \equiv 1$ for $0 \le i \le n$ then

$$J_n(t) = \frac{(t-A)^n}{n!}. \tag{12}$$

II. Main Results. The following lemma, which is Theorem (1) of Singh and Kusano [5], establishes a growth condition for solutions of equation (5).

Lemma 1. *Suppose that* (6) *holds, and there exists a number* $\gamma \in (0, 1]$ *and a continuous function* $q : [A, \infty) \to [0, \infty)$ *such that*

$$|F(t, s)| \le q(t)|s|^\gamma \quad \textit{for } (t, s) \in [A, \infty) \times R. \tag{13}$$

Suppose moreover that

$$\int^\infty p_n(t)|f(t)| dt < \infty \tag{14}$$

and

$$\int^\infty [J_{n-1}(t)]^\gamma p_n(t) q(t) dt < \infty. \tag{15}$$

Then every nontrivial solution $y(t)$ *of equation* (5) *satisfies*

$$y(t) = 0(J_{n-1}(t)) \quad \textit{as } t \to \infty.$$

Lemma 2. *The function* $u(|x|), x = (x_1, x_2, x_3)$ *is a solution of equation (1) in an exterior domain*

$$\Omega = \{x \in R^3 : |x| \ge T\}, \quad T \ge A,$$

if and only if $u(t)$ *is a solution of the ordinary differential equation*

$$\frac{1}{t}\frac{d^4}{dt^4}(tu) + g(t)u = f(t), \quad t \ge T, \quad \textit{where } t = |x|. \tag{16}$$

Proof. By direct substitution, it can be easily verified that

$$\Delta u(|x|) = \frac{1}{t^2}\frac{d}{dt}t^2\frac{du}{dt}$$
$$= \frac{1}{t}\frac{d^2}{dt^2}(tu), \quad t = |x|$$

$$\Delta^2 u(|x|) = \Delta(\frac{1}{t}\frac{d^2}{dt^2}(tu))$$
$$= \frac{1}{t}\frac{d^2}{dt^2}(t \times \frac{1}{t}\frac{d^2}{dt^2}(tu))$$
$$= \frac{1}{t}\frac{d^2}{dt^2}(\frac{d^2}{dt^2}(tu))$$
$$= \frac{1}{t}\frac{d^2}{dt^2}(t\frac{d^2u}{dt^2} + 2\frac{du}{dt})$$
$$= \frac{1}{t}(t\frac{d^4u}{dt^4} + 4\frac{d^3u}{dt^3})$$
$$= \frac{1}{t}\frac{d^4}{dt^4}(tu). \tag{17}$$

By direct substitution from (17) into equation (1), we get equation (16). Conversely if $u(t)$ is a solution of equation (16) where $t = |x|$ then we can obtain equation (1). The proof of Lemma 2 is now complete.

Theorem 1. *Suppose*

$$\int^{\infty} t|f(t)|dt < \infty \tag{18}$$

and

$$\int^{\infty} t^3|g(t)|dt < \infty \tag{19}$$

Then every symmetric solution $u(|x|)$ of equation (1) satisfies

$$u(|x|) = 0(|x|^2) \tag{20}$$

as $|x| \to \infty$.

Proof. If we choose $p_0(t) = 1/t, p_1(t) = p_2(t) = p_3(t) = 1, p_4(t) = t$, and $\gamma = 1$, then all conditions of Lemma (1) are satisfied for equation (16). Hence any solution $u(t)$ of this equation satisfies $tu(t) = 0(t^3) \Rightarrow u(t) = 0(t^2)$. The conclusion now follows by Lemma 2.

Our next lemma is Theorem (3) of our work [5] which gives us a stronger result for equation (1).

Lemma 3. *Suppose that* (6) *holds and there exists a number* $\gamma \in (0, 1]$ *and a continuous function* $q : [A, \infty) \to [0, \infty)$ *such that* (13) *holds. Suppose moreover that*

$$\int^{\infty} K_{n-1}(t)|f(t)|dt < \infty \tag{21}$$

and

$$\int^{\infty} [J_{n-1}(t)]^{\gamma} K_{n-1}(t) q(t) dt < \infty. \tag{22}$$

Then every oscillatory solution $y(t)$ *of* (5) *satisfies*

$$\lim_{t \to \infty} |y(t)|/p_0(t) = 0. \tag{23}$$

This lemma leads us to the following stronger result for equation (1).

Theorem 2. *Suppose*

$$\int^{\infty} t^4 |f(t)| dt < \infty \tag{24}$$

and

$$\int^{\infty} t^6 |g(t)| dt < \infty. \tag{25}$$

Then all oscillatory spherically symmetric solutions $u(|x|)$ *of* (1) *satisfy*

$$\lim_{|x| \to \infty} |x| u(|x|) = 0.$$

Proof. Since

$$k_3(t) = 0(t^3)$$
$$J_3(t) = 0(t^3)$$

and $\gamma = 1$, all conditions of Lemma (3) are satisfied for equation (16). Thus any oscillatory solution $u(t)$ of (16) satisfies

$$\operatorname{Lim}_{t \to \infty} u(t)/p_0(t) = 0.$$

Since $p_0(t) = \frac{1}{t}$, we have $tu(t) \to 0$ as $t \to \infty$. The conclusion of the theorem follows.

Our next theorem gives sufficient conditions for all bounded solutions of equation (1) to be oscillatory.

Theorem 3. *Suppose that for any* $T \geq A$

$$Lim \sup_{t \to \infty} \left(\frac{1}{t^3} \int_T^t (t-s)^3 s f(s) ds \right) = \infty \tag{26}$$

$$\operatorname*{Lim\,inf}_{t\to-\infty}\left(\frac{1}{t^3}\int_T^t (t-s)^3 s f(s)ds\right) = -\infty \tag{27}$$

and

$$\operatorname*{Lim}_{t\to\infty}\frac{1}{t^3}\int_T^t s|g(s)|(t-s)^3 ds < \infty. \tag{28}$$

Then all spherically symmetric bounded solutions $u(|x|)$ of equation (1) defined in an exterior domain

$$\Omega_T = \{x = (x_1, x_2, x_3) : |x| > T\}$$

are oscillatory.

Proof. We prove this for equation (16). Let $u(t)$, to the contrary, be a bounded nonoscillatory solution of (16). Without loss of generality suppose that $u(t) > 0$ for $t \geq T$. On direct integration from equation (16), we have

$$tu'''(t) - Tu'''(T) - \int_T^t u'''(s)ds + 4\int_T^t u'''(s)ds$$
$$+\int_T^t sg(s)u(s)ds = \int_T^t sf(s)ds.$$

On repeated integration three more times, we get

$$\frac{tu(t)}{t^3} - \frac{M(t)}{t^3} + \frac{1}{t^3}\int_T^t \frac{(t-s)^3}{3!} sg(s)u(s)ds$$
$$= \frac{1}{t^3}\int_T^t \frac{(t-s)^3}{3!} sf(s)ds \tag{29}$$

where

$$M(t) = (C_1 + C_2(t-T) + C_3(t-T)^2 + C_4(t-T)^3)$$

and C_1, C_2, C_3, and C_4 are appropriate constants.

Since the left side of (29) is bounded as $t \to \infty$, and the right hand side swings between $-\infty$ and ∞, a contradiction is reached. Hence by virtue of equation (16), all bounded spherically symmetric solutions $u(|x|)$ of equation (1) are oscillatory. This completes the proof.

If in Theorem (3), we require an additional condition that $g(t) > 0$ for $t \geq A$, then its proof reveals that boundedness of the solutions of (1) or of (16) is not essential. This leads us to the following stronger theorem.

Theorem 4. *In addition to the conditions of Theorem (3), suppose $g(t) \geq 0$ for $t \geq A$. Then all solutions of equation (1) continuable into the exterior domain Ω_T are oscillatory.*

Proof. In reexamination of the proof of Theorem (3), we notice that the only place where boundedness of $u(t)$ is needed is in the integral on the left side of (29). Since $g(t) \geq 0$, it is no longer needed. This observation completes the proof.

Example 1. Consider the elliptic equation

$$\Delta^2 u(|x|) + e^{-|x|}u(|x|) = |x|^5 \sin(|x|) \tag{30}$$

where

$$|x| = \left(\sum_{i=1}^{3} x_i^2\right)^{\frac{1}{2}}.$$

Then the corresponding companion differential equation is

$$\frac{1}{t}\frac{d^4(ut)}{dt^4} + \epsilon^{-t}u = t^5 \sin t, \quad t > 0 \tag{31}$$

which satisfies all the conditions of Theorem (3). Hence all solutions of equation(30) continuable beyond the hypersphere

$$\sum_{i=1}^{3} x_i^2 = T^2, \quad T > A.$$

are oscillatory.

Remark 1. The proof of Theorem (3) reveals that if in addition to (26), (27), $f(t)$ is also slowly oscillating then bounded solutions of equations (16) and (1) will also be slowly oscillating. We state this fact as Theorem (4).

Theorem 5. *Suppose that* (26), (27) *and* (28) *hold. Further suppose that for any bounded function* $L(t)$ *the funcition* $f(t) - L(t)g(t)$ *is slowly oscillating. Then all bounded solutions of equations* (16) *and* (1) *are slowly oscillating.*

Proof. Let $u(t)$ be a bounded solution of equation (16). Then by Theorem (3), $u(t)$ is oscillatory. The oscillation of $u(t)$ implies oscillation of $u^{IV}(t)$. From equation (16)

$$\frac{d^4}{dt^4}(tu(t)) = t(f(t) - u(t)g(t)). \tag{32}$$

Since $u(t)$ is bounded, (32) implies that

$$\frac{d^4}{dt^4}(tu(t))$$

is also slowly oscillating.

We now observe an elementary fact that if the derivative of an oscillatory function $h(t)$ is slowly oscillating so is the function $h(t)$. To see this we note that if $h(t_1) = h(t_2) = 0$, $h(t) \neq 0$ for $t \in (t_1, t_2)$ then

$$\int_{t_1}^{t_2} h'(t)dt = 0 \tag{33}$$

Thus between any two consecutive zeros of $h(t)$, there must be a zero of $h'(t)$. Hence if $h'(t)$ is slowly oscillating, so is $h(t)$. This observation leads to the fact that slow oscillation of

$$\frac{d^4}{dt^4}(tu(t))$$

implies slow oscillation of

$$\frac{d^3}{dt^3}(tu(t)), \frac{d^2}{dt^2}(tu(t)), \frac{d}{dt}(tu(t))$$

and consequently of $tu(t)$. This completes the proof of Theorem 4.

Remark 2. Our next example shows that conditions of Theorem (4) are quite practical.

Example 2. Consider the equation

$$\Delta^2 u(|x|) + \frac{1}{|x|^{10}} \sin(|x|)u(|x|) = \sin(\log(|x|)) \tag{34}$$

satisfies the conditions and conclusion of this theorem. This follows from the fact that for the companion equation

$$\frac{1}{t}\frac{d^4}{dt^4}(tu(t)) + \frac{1}{t^{(10)}} \sin(t)u = \sin(\log(t)), \quad t > 0 \tag{35}$$

the function

$$\frac{1}{t^{10}}(t^{10} \sin(\log(t)) - \sin(t)u(t))$$

is slowly oscillating for any bounded $u(t)$.

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