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NOTE ON AN INTEGRAL INEQUALITY FOR CONCAVE FUNCTIONS

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Abstract. We prove: Let $p \in C^2[a,b]$ be non-negative and concave, and let $f \in C^2[a,b]$ with $f(a) = f(b) = 0$. Then

$$\left(\int_a^b p(x)(f'(x))^2 dx\right)^2 \leq \int_a^b p(x)(f(x))^2 dx \int_a^b p(x)(f''(x))^2 dx.$$

Moreover, we determine all cases of equality.

In a recently published article L. -C. Shen [3] presented some interesting integral inequalities involving polynomials. To prove his results Shen made use of an identity which we state here in a slightly modified form.

Lemma. *Let $p \in C^2[a,b]$ and $f \in C^2[a,b]$. Then we have for all positive real numbers $\mathcal{L}$:*

$$2\int_a^b p(x)(f'(x))^2 dx = \int_a^b [\mathcal{L}p(x) + p''(x)](f(x))^2 dx$$
$$+\frac{1}{\mathcal{L}}\int_a^b p(x)(f''(x))^2 dx - \frac{1}{\mathcal{L}}\int_a^b p(x)[\mathcal{L}f(x) + f''(x)]^2 dx + C, \tag{1}$$

where

$$C = 2p(b)f(b)f'(b) - 2p(a)f(a)f'(a) + p'(a)(f(a))^2 - p'(b)(f(b))^2.$$

It is the aim of this note to show how the Lemma can be applied to establish a new integral inequality for concave functions which is related to the classical inequality

$$\left(\int_0^\infty (f'(x))^2 dx\right)^2 \leq 4\int_0^\infty (f(x))^2 dx \int_0^\infty (f''(x))^2 dx,$$

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due to G.H. Hardy, J.E. Littlewood and G. Pólya [2, p.187].

Theorem. *Let $p \in C^2[a,b]$ be non-negative and concave, and let $f \in C^2[a,b]$ with $f(a) = f(b) = 0$. Then*

$$\left(\int_a^b p(x)(f'(x))^2 dx\right)^2 \leq \int_a^b p(x)(f(x))^2 dx \int_a^b p(x)(f''(x))^2 dx. \tag{2}$$

If $p \not\equiv 0$ and $f \not\equiv 0$, then the sign of equality holds in (2) if and only if p is linear and $f(x) = A\cos\frac{k\pi x}{b-a} + B\sin\frac{k\pi x}{b-a}$, where $A, B \in \mathbb{R}$, $(A,B) \neq (0,0)$, and $k \in \mathbb{N}$ such that $A\cos\frac{k\pi a}{b-a} + B\sin\frac{k\pi a}{b-a} = 0$.

Proof. Let $p \not\equiv 0$ and $f \not\equiv 0$. Since $f(a) = f(b) = 0$ we conclude that $f'' \not\equiv 0$, which implies

$$\mathcal{L}_0 = \left(\int_a^b p(x)(f''(x))^2 dx \Big/ \int_a^b p(x)(f(x))^2 dx\right)^{1/2} > 0.$$

If we replace in (1) $\mathcal{L}$ by $\mathcal{L}_0$, then we get

$$\begin{aligned} &2\int_a^b p(x)(f'(x))^2 dx \\ =&2\left(\int_a^b p(x)(f(x))^2 dx \int_a^b p(x)(f''(x))^2 dx\right)^{1/2} \\ &+\int_a^b p''(x)(f(x))^2 dx - \frac{1}{\mathcal{L}_0}\int_a^b p(x)[\mathcal{L}_0 f(x) + f''(x)]^2 dx. \end{aligned} \tag{3}$$

Since p is concave we have $p'' \leq 0$, so that (3) leads to inequality (2).

We discuss the cases of equality. A simple calculation reveals that the sign of equality holds in (2) if $p(x) = c_0 + c_1 x$ and $f(x) = A\cos\frac{k\pi x}{b-a} + B\sin\frac{k\pi x}{b-a}$, where $A, B \in \mathbb{R}$, $k \in \mathbb{N}$ satisfy A $\cos\frac{k\pi a}{b-a} = -B\sin\frac{k\pi a}{b-a}$.

If equality is valid in (2), then we obtain from (3) that

$$\int_a^b p(x)[\mathcal{L}_0 f(x) + f''(x)]^2 dx = 0 \tag{4}$$

and

$$\int_a^b p''(x)(f(x))^2 dx = 0. \tag{5}$$

To determine f we conclude from (4) that

$$\mathcal{L}_0 f(x) + f''(x) = 0,$$

which leads to

$$f(x) = A\cos(\sqrt{\mathcal{L}_0}x) + B\sin(\sqrt{\mathcal{L}_0}x); \quad A, B \in \mathbb{R}.$$

From $f(a) = f(b) = 0$ we get

$$\begin{aligned} A\cos(\sqrt{\mathcal{L}_0}a) + B\sin(\sqrt{\mathcal{L}_0}a) &= 0 \\ A\cos(\sqrt{\mathcal{L}_0}b) + B\sin(\sqrt{\mathcal{L}_0}b) &= 0. \end{aligned} \tag{6}$$

Since $(A, B) \neq (0, 0)$ we conclude from (6) that

$$\sin(\sqrt{\mathcal{L}_0}(b-a)) = 0,$$

which implies

$$\sqrt{\mathcal{L}_0} = k\pi/(b-a), \quad k \in \mathbb{N}.$$

Thus, we have

$$f(x) = A\cos\frac{k\pi x}{b-a} + B\sin\frac{k\pi x}{b-a}; \quad A, B \in \mathbb{R}, \quad (A, B) \neq (0, 0), \quad k \in \mathbb{N}, \tag{7}$$

with

$$A\cos\frac{k\pi a}{b-a} + B\sin\frac{k\pi a}{b-a} = 0.$$

And, from (5) and (7) we obtain $p'' \equiv 0$, which implies that p is linear.

Remarks. 1) Inequality (2) is in general not true if the assumption "p is concave" will be dropped. For instance, if we set $a = 0$, $b = \pi$, $p(x) = x^2$ and $f(x) = \sin(x)$, then (2) holds with " $>$ " instead of " $\leq$ ".
2) Concerning further integral inequalities for concave functions we refer to [1] and the references therein.

References

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