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SOME PROPERTIES OF FIBONACCI LANGUGAES

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Abstract. Two particular types of Fibonacci languages $F^1_{a,b} = \{a, b, ab, bab, abbab, \ldots\}$ and $F^0_{a,b} = \{a, b, ba, bab, babba, \ldots\}$ were defined on the free monoid X^* generated by the alphabet $X = \{a, b\}$. In this paper we investigate some algebraic properties of these two types of Fibonacci languages. We show that a general Fibonacci language is a homomorphical image of either $F^1_{a,b}$ of $F^0_{a,b}$. We also study the properties of Fibonacci language related to formal language theory and codes. We obtained the facts that every Fibonacci word is a primitive word and for any $u \in X^+$, u^4 is not a subword of any words in both $F^1_{a,b}$ and $F^0_{a,b}$.

1. Introduction

This paper is a study of some algebraic properties of Fibonacci languages. Let X^* be the free monoid generated by an alphabet X consisting of exactly two letters, i.e., $X = \{a, b\}$. Any element of X^* is called a *word* over X and any subset of X^* is called a *language* over X. Let $X^+ = X^* \backslash \{1\}$, where 1 is the empty word. The length of a word $u \in X^*$ is denoted by $lg(u)$.

Let $w_1 = u, w_2 = v$, where $u, v \in X^+$. The two types of Fibonacci sequences are defined recursively as follows:

(1) $w_1 = u,\ w_2 = v,\ w_3 = uv, \ldots, w_n = w_{n-2}w_{n-1}, w_{n+1} = w_{n-1}w_n, \ldots;$

(2) $w_1 = u,\ w_2 = v,\ w_3 = vu, \ldots, w_n = w_{n-1}w_{n-2}, w_{n+1} = w_n w_{n-1}, \ldots.$

We note that for $n \geq 4$, $2 \cdot lg(w_{n-2}) < lg(w_n) < 2 \cdot lg(w_{n-1})$.

Let $F^1_{u,v}$ be the set formed by taking the union of the first sequence and $F^0_{u,v}$ be the set formed by taking the union of the second sequence respectively. That is

$$F^1_{u,v} = \{u, v, uv, vuv, uvvuv, \ldots\}$$

and

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$$F^0_{u,v} = \{u, v, vu, vuv, vuvvu, \ldots\}.$$

We will call $F^1_{u,v}$ a Fibonacci language of type 1 and $F^0_{u,v}$ a Fibonacci language of type 0 respectively. If we take the initial words w_1 and w_2 to be the letters a and b, then we get the two types of Fibonacci languages

$$F^1_{a,b} = \{a, b, ab, bab, abbab, \ldots\} \quad \text{and} \quad F^0_{a,b} = \{a, b, ba, bab, babba, \ldots\}.$$

These Fibonacci languages will be the simplest ones and will be called the *1-atom Fibonacci language* and *0-atom Fibonacci language* respectively. We will see that any other Fibonacci languages will related to one of these two atom Fibonacci languages. For convenience we will consider a Fibonacci language also as a sequence. In order to discuss properties on languages, we need to define some terms which we need. For any two languages $A, B \subseteq X^*$, let $AB = \{xy | x \in A, y \in B\}$ and $A^* = A^0 \cup A \cup A^2 \cup A^3 \cup \cdots$, where $A^0 = \{1\}$. If u is a word such that $u = xwy$, where $w \in X^+, x, y \in X^*$, then the word w will be called a *subword* of u. A subword w of u is a *proper subword* of u if $u = xwy$ such that $x, y \in X^+$. For $u \in X^+$, let $E(u)$ and $\bar{E}(u)$ be the set of all subwords of u and the set of all proper subwords of u respectively. A word $u \in X^+$ is called a *primitive word* if it can not be written as a power of any other word. We will let Q be the set of all primitive words over X. And for $n \geq 2$, we let $Q^{(n)} = \{f^n | f \in Q\}$. In particular $Q^{(1)} = Q$. We call the word u an *overlapping word* if u is such that $u = wx = yw$ for some $w \in X^+$, $x, y \in X^+$. Let $u, v \in X^+$. We say that v is a *conjugate* of u if $u = xy$, for some $x, y \in X^*$, then $v = yx$. It is easy to see that if v is a conjugate of u, then u is a conjugate of v. Thus we may call two words with such a property a *conjugate pair*. It is known that $xy \in Q^{(n)}$ if and only if $yx \in Q^{(n)}$ for any $n \geq 1$([3]). A *square free word* is a word such that every subword of u is primitive. A word u is *cubic-free* if no subword of u is x^3 for some $x \in X^+$. We let G be the set of all square free words over X. If $X = \{a, b\}$, then $G = \{a, b, ab, ba, aba, bab\}$. Let $G^{(1)} = G_1 = G$ and also for any $n \geq 2$, let the set $G_n = \{u \in X^+ | E(u) \cap Q^{(n)} \neq \emptyset\}$ and $G^{(n)} = G_n \setminus G_{n+1}$. Then clearly,

$$X^+ = G^{(1)} \cup G^{(2)} \cup G^{(3)} \cup \cdots,$$

and the union is a disjoint union. We note that the set $G^{(1)} \cup G^{(2)}$ consists of all cubic-free words over X. A language $L \subseteq X^+$ is a code if L generates a free submonoid of X^*. A 2-code A is a language such that every two elements from A is a code.

In this paper we investigate mostly the algebraic properties of the two types of atom Fibonacci languages. We show that every atom Fibonacci word is a primitive word. It is proved that the two types of atom Fibonacci words of the same length is a conjugate pair. Both $F^1_{a,b}$ and $F^0_{a,b}$ are 2-codes and each partitioned into two infinite subsets F^1_1, F^1_2 and F^0_1, F^0_2, all of them are codes. A language $L \subseteq X^*$ is *regular* if L can be recognized by an automaton or equivalently the index of the principal congruence determined by the language L is finite. We also obtained the fact that both atom Fibonacci languages are regular free languages in the sense that every infinite subset of it is not a regular language. There are some atom Fibonacci words which are not cubic free. We obtained that for any $u \in X^+$, u^4 is not a subword of any atom Fibonaccci words. The terminologies not defined in here please refer to [3].

2. Elementary Properties of the Languages $F^0_{a,b}$ and $F^1_{a,b}$

A mapping h from X^+ into X^+ is called a homomorphism if $h(uv) = h(u)h(v)$ for all $u, v \in X^+$. For a given homomorphism $h : X^+ \to X^+$ and for any language $L \subseteq X^+$, we let $h(L) = \{h(u) \mid u \in L\}$. The substitution of the form, for $x, y \in X^+$,

$$a \to x, b \to y$$

induces a homomorphism from X^+ into X^+. For example, if we let $h : X^+ \to X^+$ by $h(a) = u, h(b) = v$, $u, v \in X^+$ be two fixed words, then h is clearly a homomorphism. Thus the following lemma is immediate.

Lemma 2.1. *Every Fibonacci sequence is a homomorphic image of the atom Fibonacci sequences $F^1_{a,b}$ or $F^0_{a,b}$.*

Due to above fact, some algebraic properties of the atom Fibonacci languages can be carried through by a homomorphism. The study of the algebraic properties on the atom Fibonacci languages is then important.

In [2] de Luca has proved that every word in $F^0_{a,b}$ is a primitive word. In here we give a different proof to show that words in both $F^1_{a,b}$ and $F^0_{a,b}$ are primitive words. In order to do this we need the following known result. Recall that a non-empty word f is a primitive word if f is not a power of any other word.

Lemma 2.2. ([4], [5]) *Let a, c be two primitive words. If $a^m = c^k x$ ($a^m = xc^k$), $m, k \geq 2$ and x is a prefix (suffix) of c, then $a = c$.*

Proposition 2.3. *The atom Fibonacci languages $F^0_{a,b}$ and $F^1_{a,b}$ are subsets of Q.*

Proof. Consider $F^0_{a,b} = \{a, b, ba, bab, babba, babbabab, \cdots, w_n, w_{n+1}, w_{n+2}, \cdots\}$. We show that every word in $F^0_{a,b}$ is a primitive word by using the above lemma. The proof will be completed by induction on terms. We observe that the first few terms in $F^0_{a,b}$ are primitive words. Now we assume that $w_1, w_2, \ldots, w_{n-1}, w_n$ are primitive words and we show that, for $n \geq 3$, the term w_{n+1} is also a primitive word. We note that $w_n = w_{n-1}w_{n-2}$ and

$$w_{n+1} = w_{n-1}w_{n-2}w_{n-1}.$$

Since xy is a primitive word if and only if yx is a primitive word, (Proposition 1.11 [3]), to show that w_{n+1} is a primitive word is equivalent to show that the word

$$u = (w_{n-1})^2 w_{n-2}$$

is primitive. Suppose the word $u = f^m, m \geq 2, f$ is a primitive word. Since w_{n-2} is a prefix of w_{n-1} in our sequence, by the above lemma, we see that $f = w_{n-1}$, which is not possible. Thus the word u is primitive. Therefore w_{n+1} is a primitive word. By induction conclusion, we have that every Fibonacci word in $F^0_{a,b}$ is a primitive word.

For the case of $F^1_{a,b}$, since each Fibonacci word w_n is a suffix of w_{n+1}, by a similar argument and with a help of the above lemma, we can prove that every Fibonacci word in $F^1_{a,b}$ is a primitive word.

Proposition 4.9 in section 4 generalized the present result.

3. Conjugate Property of two Types of Fibonacci Words

In the two types of atom Fibonacci languages $F^0_{a,b}$ and $F^1_{a,b}$, the corresponding terms of Fibonacci words are of the same length. In the following, we show that they are in fact a conjugate pair. For convenience, we use the notation $u \leq_p v$ to mean that the word u is a prefix of the word v and $u \leq_s v$ to mean that the word u is a suffix of the word v.

To prove the following, we let

$$F^0_{a,b} = \{a, b, ba, bab, babba, \ldots\} = \{z_1, z_2, z_3, z_4, z_5, \ldots\}; \tag{3.1}$$

$$F^1_{a,b} = \{a, b, ab, bab, abbab, \ldots\} = \{w_1, w_2, w_3, w_4, w_5, \ldots\}. \tag{3.2}$$

Lemma 3.1. *For $n \geq 1$, let z_n, w_n be the n-th words in $F^0_{a,b}$ and $F^1_{a,b}$ respectively. For $n \geq 3$, let $z_n = z'_n z''_n$, $w_n = w'_n w''_n$, where $z'_n, z''_n, w'_n, w''_n \in X^*$ with $lg(z''_n) = lg(w'_n) = 2$. Then*

$$z'_n = w''_n.$$

Proof. First, by observation for the n-th terms z_n in $F^0_{a,b}$ and w_n in $F^1_{a,b}$, we have that $lg(z_n) = lg(w_n)$ and $z_3 = ba$, $z_4 = bab$, $w_3 = ab$, $w_4 = bab$. Clearly, $ba \leq_s z_3$, $ab \leq_s z_4$, $ab \leq_p w_3$, $ba \leq_p w_4$. Since $z_{n+2} = z_{n+1}z_n$; $w_{n+2} = w_n w_{n+1}$, clearly for $n \geq 3$, we have that, for some $z'_n, w''_n \in X^*$,

$$z_n = \begin{cases} z'_n ba, & \text{if n is odd;} \\ z'_n ab, & \text{if n is even;} \end{cases} \tag{3.3}$$

$$w_n = \begin{cases} abw''_n, & \text{if n is odd;} \\ baw''_n, & \text{if n is even.} \end{cases} \tag{3.4}$$

Our proof will be completed by induction on terms $n \geq 3$.

(1) It is clear that for $n = 3$, $z_3 = ba = (1)(ba), w_3 = ab = (ab)(1), z'_3 = 1 = w''_3$; for $n = 4$, $z_4 = (b)(ab), w_4 = (ba)(b)$, $z'_4 = b = w''_4$. Thus the lemma is true for $n = 3$ and 4.

(2) Suppose $z'_n = w''_n$, for $3 \leq n \leq m$.
(3) Let $n = m + 1$. We consider the following two cases:
  (a) If m is odd, then $n = m + 1$ is even, by Equations (3.3) and (3.4) and by induction hypothesis,

$$\begin{aligned} z_n = z_{m+1} &= z_{m-1}z_m \\ &= z_{m-1}z_{m-2}z_{m-1} \\ &= z'_{m-1}abz'_{m-2}baz'_{m-1}ab \\ &= w''_{m-1}abw''_{m-2}baw''_{m-1}ab \\ &= w''_{m-1}w_{m-2}w_{m-1}ab \\ &= w''_{m-1}w_m ab. \end{aligned}$$

On the other hand, by Equation (3.3), $z_{m+1} = z'_{m+1}ab$, we have then $z'_{m+1} = w''_{m-1}w_m$. That is,

$$z'_n = w''_{n-2}w_{n-1}. \tag{3.5}$$

Similarly, by Equation (3.4),

$$\begin{aligned} w_{m+1} &= w_{m-1}w_m \\ &= baw''_{m-1}w_m. \end{aligned}$$

Again by Equation (3.4), $w_{m+1} = baw''_{m+1}$, then we have that $w''_{m+1} = w''_{m-1}w_m$. That is,

$$w''_n = w''_{n-2}w_{n-1}. \tag{3.6}$$

Now by Equations (3.5) and (3.6), $z'_n = w''_n$.
  (b) If m is even, then $m + 1$ is odd. By a similar argument as in the case (a), we can show that $z'_n = w''_n$.

Summing up Cases (a), (b) and by induction conclusion, we have that $z'_n = w''_n$, for $n \geq 3$.

Lemma 3.2. *For $n \geq 3$, let $z_n = z'_n z''_n, w_n = w'_n w''_n$, where $z_n, w_n, z'_n, z''_n, w'_n, w''_n$ are the same definitions as in Lemma 3.1. Then*

$$z''_{n+1}z_n = w_n w'_{n+1}.$$

Proof. First, by Equations (3.3) and (3.4), we have

$$z''_n = \begin{cases} ba, & \text{if n is odd;} \\ ab, & \text{if n is even.} \end{cases} \tag{3.7}$$

$$w'_n = \begin{cases} ab, & \text{if n is odd;} \\ ba, & \text{if n is even.} \end{cases} \tag{3.8}$$

Now we consider following two cases:

(1) If $n = 2k+1, k \geq 1$, then, by Lemma 3.1 and Equations (3.3), (3.4), (3.7), (3.8),

$$\begin{aligned} z''_{n+1} z_n &= abz_{2k+1} \\ &= abz'_{2k+1} ba \\ &= abw''_{2k+1} ba \\ &= w_{2k+1} ba \\ &= w_n w'_{n+1}. \end{aligned}$$

(2) If $n = 2k+2, k \geq 1$, then, by the similar argument as in (1), we have that $z''_{n+1} z_n = w_n w'_{n+1}$.

By Cases (1) and (2), then for all $n \geq 3$, $z''_{n+1} z_n = w_n w'_{n+1}$.

Proposition 3.3. *Two Fibonacci words in $F^0_{a,b}$ and $F^1_{a,b}$ of the same length form a conjugate pair.*

Proof. Consider Equations (3.1) and (3.2). By observation, we have that $z_1 = w_1, z_2 = w_2, z_3 = ba, w_3 = ab$ and $z_4 = w_4$. Thus, for $n \leq 4$, z_n and w_n form a conjugate pair.

We now show that for $n \geq 5$, z_n and w_n form a conjugate pair. By Lemmas 3.1 and 3.2, we have that, for $n \geq 3$,

$$\begin{aligned} z_{n+2} &= z_{n+1} z_n \\ &= z'_{n+1} z''_{n+1} z_n \\ &= w''_{n+1} w_n w'_{n+1} \end{aligned}$$

and

$$\begin{aligned} w_{n+2} &= w_n w_{n+1} \\ &= w_n w'_{n+1} w''_{n+1}. \end{aligned}$$

It is now clear that z_{n+2} and w_{n+2} form a conjugate pair. We then conclude that for $n \geq 5$, z_n and w_n form a conjugate pair.

4. Fibonacci Languages Related to Codes and Formal Language Theory

From Lemma 1.7 ([4]) and Proposition 1.25 ([4]), we have that every non-empty subset of Q is a 2-code. The following lemma follows from Proposition 2.3.

Lemma 4.1. *The atom Fibonacci languages $F^0_{a,b}$ and $F^1_{a,b}$ are 2-codes.*

Lemma 4.2. (1) *For Fibonacci words w_n, w_{n+2}, w_{n+3} in $F^1_{a,b}$, we have $w_{n+2} w_{n+2} = w_n w_{n+3}$.* (2) *For Fibonacci words z_n, z_{n+2}, z_{n+3} in $F^0_{a,b}$, we have $z_{n+2} z_{n+2} = z_{n+3} z_n$.*

Proof. (1) Since $w_{n+3} = w_{n+1}w_{n+2}$, we have

$$\begin{aligned} w_n w_{n+3} &= w_n(w_{n+1}w_{n+2}) \\ &= (w_n w_{n+1})w_{n+2} \\ &= w_{n+2}w_{n+2}. \end{aligned}$$

(2) By a similar way, we can show that (2) holds true.

From the sequences $F^0_{a,b}$ and $F^1_{a,b}$, we see that for any $n \geq 1$, the set $\{w_n, w_{n+1}, w_{n+2}\}$ is not a code. From the above lemma, we see that, for $n \geq 1$, the set $\{w_n, w_{n+2}, w_{n+3}\}$ is not a code. Moreover, for any code $A \subset F^1_{a,b}$ such that $w_m, w_{m+1} \in A$, for some $m \geq 1$, then, clearly, $w_{m+i} \notin A$, for all $i \geq 2$. Thus if a subset A of $F^1_{a,b}$ is a code and contains two consecutive Fibonacci words, then the set A must be finite. Nevertheless, we will show that $F^1_{a,b}$ and $F^0_{a,b}$ contain infinite subsets which are codes. For constructing such infinite subsets, we just do by partitioning the two types of atom Fibonacci languages. First we need the following

Proposition 4.3. ([4]) *Let $A \subseteq X^+$. Then A is a code if and only if $x_1x_2\cdots x_n = y_1y_2\cdots y_n$, $x_i, y_i \in A, i = 1, 2, \ldots, n$ implies $x_i = y_i, i = 1, 2, \ldots, n$.*

Let

$$F^0_{a,b} = \{a, b, ba, bab, babba, \ldots\} = \{z_1, z_2, z_3, z_4, z_5, \ldots\}; \tag{4.1}$$

$$F^1_{a,b} = \{a, b, ab, bab, abbab, \ldots\} = \{w_1, w_2, w_3, w_4, w_5, \ldots\}. \tag{4.2}$$

and let

$$\begin{aligned} F^1_1 &= \{w_n \in F^1_{a,b} | n = 1, 3, 5, 7, \ldots\} \\ &= \{w_1, w_3, w_5, w_7, \ldots\} \\ &= \{a, ab, abbab, abbabbababbab, \ldots\}; \end{aligned} \tag{4.3}$$

$$\begin{aligned} F^1_2 &= \{w_n \in F^1_{a,b} | n = 2, 4, 6, 8, \ldots\} \\ &= \{w_2, w_4, w_6, w_8, \ldots\} \\ &= \{b, bab, bababbab, bababbababbabbababbab, \ldots\}; \end{aligned} \tag{4.4}$$

$$\begin{aligned} F^0_1 &= \{z_n \in F^0_{a,b} | n = 1, 3, 5, 7, \ldots\} \\ &= \{z_1, z_3, z_5, z_7, \ldots\} \\ &= \{a, ba, babba, babbababbabba, \ldots\}; \end{aligned} \tag{4.5}$$

$$\begin{aligned} F^0_2 &= \{z_n \in F^0_{a,b} | n = 2, 4, 6, 8, \ldots\} \\ &= \{z_2, z_4, z_6, z_8, \ldots\} \\ &= \{b, bab, babbabab, babbababbabbababbabab, \ldots\}. \end{aligned} \tag{4.6}$$

At this stage we like to give the following remarks:

Remark.

(1) No words in F_1^1 is a prefix of any words in F_2^1 and also no words in F_2^1 is a prefix of any words in F_1^1.

(2) Every n-th word in F_1^1 is a prefix and also a suffix of $(n+1)$-th word in F_1^1, $n \geq 2$.

(3) Every n-th word in F_2^1 is a prefix and also a suffix of $(n+1)$-th word in F_2^1, $n \geq 1$.

We claim that the four subsets F_1^1, F_2^1, F_1^0 and F_2^0 are codes. First, we prove the following some lemmas.

Lemma 4.4. *Let F_1^1 be defined by Equation (4.3). If $w_n \in F_1^1, n \geq 1$, and $w_n = x_1x_2$, for $x_1 \in F_1^1$ and $x_2 \in X^+$, then $b \leq_p x_2$.*

Proof. Let $w_n \in F_1^1$. Then n is odd. We have

$$\begin{aligned} w_n &= (w_{n-2})w_{n-1} \\ &= (w_{n-4})w_{n-3}w_{n-1} \\ &= (w_{n-6})w_{n-5}w_{n-3}w_{n-1} \\ &= \cdots \\ &= (w_1)w_2w_4\cdots w_{n-1}. \end{aligned}$$

In each expression above for w_n, we see that

$$w_{n-2},\ w_{n-4},\ w_{n-6},\ \cdots,\ w_1$$

are in F_1^1, and $w_2,\ w_4,\ \ldots,\ w_{n-3},\ w_{n-1}$ are in F_2^1. It is now clear that $b \leq_p x_2$.

The following is a dual case of the above lemma.

Lemma 4.5. *Let F_2^1 be defined by Equation (4.4). If $w_n \in F_2^1, n \geq 1$ and $w_n = x_1x_2$, for $x_1 \in F_2^1$, $x_2 \in X^+$, then $a \leq_p x_2$.*

Corollary 4.6. *Let F_1^0, F_2^0 be defined by Equation (4.5), (4.6), respectively If $z_n, n \geq 1$, is in the set F_1^0 (F_2^0) and $z_n = x_1x_2$, for $x_1 \in F_1^0$ (F_2^0), $x_2 \in X^+$, then x_1, x_2 do not have the same suffixes.*

Proposition 4.7. *The languages F_1^1, F_2^1, F_1^0 and F_2^0 are codes.*

Proof. We only prove that F_1^1 is a code and the proofs of the rest are similar.

By Proposition 4.3, we consider $x_1x_2\cdots x_n = y_1y_2\cdots y_n$, for $n \geq 1$ and $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_n \in F_1^1$. We will show that $x_i = y_i, i = 1, 2, \ldots, n$. Our proof will be completed by induction on n. The proposition holds true for the case $n = 1$ and we assume that the proposition is true for all $n < k$.

Now let $n = k$ and suppose $x_1x_2\cdots x_n = y_1y_2\cdots y_n$. First, we show that $x_1 = y_1$. Suppose on the contrary $x_1 \neq y_1$ and no loss of generality, we assume that $lg(x_1) \geq lg(y_1)$. (i) If $lg(x_1) = lg(y_1)$, then $x_1 = y_1$. (ii) If $lg(x_1) > lg(y_1)$, then we let $x_1 = y_1y_2\cdots y_{i-1}y_i'$, for some $1 \leq i < n$, $y_i' \in X^+$, $y_i = y_i' y_i''$, $y_i'' \in X^*$. If $i \geq 2$, then since $x_1, y_1 \in F_1^1$, by Lemma 4.4, $b \leq_p y_2$. While since $y_2 \in F_1^1, a \leq_p y_2$, a contradiction. Hence $x_1 = y_1$. Thus, by both Cases (i) and (ii), $x_1 = y_1$ and $x_2 x_3 \cdots x_n = y_2 y_3 \cdots y_n$. Now by induction hypothesis, we have $x_i = y_i, i = 2, 3, \ldots, n$. Hence $x_i = y_i, i = 1, 2, 3, \ldots, n$. By induction conclusion, we have that F_1^1 is a code.

We are now in the position to discuss some language properties of the homomorphical image of the atom Fibonacci language $F_{a,b}^0$ by using the following known result. A code $L \subseteq X^+$ is a *pure code* means that for any $x \in L^*$, the primitive root of x, $\sqrt{x}$ is in L^* (see [4]).

Proposition 4.8. ([4]) *Let $h : X^* \to X^*$ be an injective homomorphism. If $h(X)$ is a pure code, then h preserves the primitive words.*

Proposition 4.9. *Let $h : X^+ \to X^+$ be an injective homomorphism such that $h(a) = u$, $h(b) = v$, where $a, b \in X$. Then the Fibonaccci language $F_{u,v}^1 = h(F_{a,b}^1)$ is a subset of Q if $\{u, v\}$ is a pure code.*

Proof. A direct consequence of the above proposition.

The set $\{u, v\}$ in the above proposition will be called the core of the Fibonacci language $F_{u,v}^1$. In the following, we consider the Fibonacci languages which are homomorphic images of $F_{a,b}^1$ in which the set $\{h(a) = u, h(b) = v\}$ is not a code.

Proposition 4.10. *Let $F_{u,v}^1$ be a Fibonacci language such that $h(a) = u, h(b) = v$. Then $F_{u,v}^1 \subseteq f^+$, for some primitive word f, if and only if $\{u, v\}$ is not a code.*

Proof. ($\Rightarrow$) Suppose for some primitive word f such that $F_{u,v}^1 \subseteq f^+$. Then clearly, $u, v \in f^+$ and $\{u, v\}$ is not a code.

($\Leftarrow$) Assuming that $\{u, v\}$ is not a code. Then by Lemma 1.25 of [4], $uv = vu$. By Lemma 1.7 of [4] the words u and v are powers of a common primitive word. Let the primitive word be f. Then clearly the Fibonacci language $F_{u,v}^1$ is a subset of f^+.

Remark. Every atom Fibonacci word is an overlapping word except for the first three terms.

Proposition 4.11. ([4]) *Let $h : X^* \to X^*$ be a homomorphism. Then the following are equivalent:*
(1) *h is injective;*
(2) *$h(X)$ is a code and $|h(X)| = |X|$;*
(3) *h preserves codes;*
(4) *h preserves 2-codes.*

We note that two-element set is a 2-code if and only if it is a code.

Proposition 4.12. *The Fibonacci core $\{u,v\}$, $u \neq v$, is a code if and only if the Fibonacci language $F^1_{u,v}$ is a 2-code.*

Proof. ($\Rightarrow$) The atom Fibonacci language $F^1_{a,b}$ is a 2-code, and $F^1_{u,v}$ is an injective homomorphical image of $F^1_{a,b}$, by the above proposition, the conclusion holds.

($\Leftarrow$) Since $F^1_{u,v}$ is a 2-code, the first two Fibonacci word $\{u,v\}$ forms a 2-code and hence $\{u,v\}$ is a code.

We remark here that if $\{u,v\}$ is a code, h an injective homomorphism and $h(a) = u, h(b) = v$, then the homomorphical images $h(F^1_1)$ and $h(F^1_2)$ of F^1_1 and F^1_2, respectively, are codes. That is both the sets $\{h(w_n)|n = 2,4,6,8,\ldots\}$ and $\{h(w_n)|n = 1,3,5,7,\ldots\}$ are codes.

In the following we consider the atom Fibonacci languages related to formal language theory. We will see that both the two languages $F^0_{a,b}$ and $F^1_{a,b}$ contain no infinite regular subsets. In other words, both $F^0_{a,b}$ and $F^1_{a,b}$ are regular free.

For a given language $L \subseteq X^*$, the principal congruence P_L determined by L is defined as follows: for any $u,v \in X^*$,

$$u \equiv v(P_L) \iff (xuy \in L \iff xvy \in L \text{ for all } x,y \in X^*).$$

In formal language theory, we call the language $L \subseteq X^*$ *regular* if the index of P_L is finite. An infinite language L is a *regular free language* if any infinite subset of L is not a regular language. We now show the following:

Proposition 4.13. *The language $F^1_{a,b}$ ($F^0_{a,b}$) is regular free.*

Proof. We show that $F^1_{a,b}$ contains no infinite regular subsets and the proof of the other case is similar. Let $A = \{u_1,u_2,u_3,\ldots\} \subseteq \{w_1,w_2,w_3,\ldots\}$ be an infinite subset of $F^1_{a,b}$. We show that for any $u_i,u_j \in A, i<j$,

$$u_i \not\equiv u_j(P_A).$$

This is true, for if $u_i = w_r, u_j = w_s$, then $r \neq s$. Let $x,y \in X^*$ be such that $xu_iy = xw_ry \in A$ with the length of xy is minimal. Then by the fact that the length of Fibonacci words are Fibonacci numbers, clearly the word xu_jy is not in A. This shows that $u_i \not\equiv u_j(P_A)$ if $i \neq j$. It follows that the index of P_A is infinite and A is not regular.

5. The Problem of Repetitive Subwords in $F^1_{a,b}$

Lemma 5.1 *Let $\{u,v\} \subset Q$ be a conjugate pair and let $u = xy$, $v = yx$, for some $x,y \in X^+$. Then $u \neq v$.*

Proof. Suppose on the contrary $u = v$. Then $xy = yx$, for $x,y \in X^+$. By Lemma 1.7([4]), we have that x and y are powers of a common word. It follows that $u,v \notin Q$, a contradiction. Thus $u \neq v$ must be true.

Proposition 5.2. *For $n \geq 1$, let z_n, w_n be the n-th words in $F^0_{a,b}$ and $F^1_{a,b}$ respectively. Then $z_n \neq w_n$, except $n = 1, 2, 4$.*

Proof. The situation is clear for the first five terms. By Proposition 3.3, for $n \geq 6$, $\{z_n, w_n\}$ is a conjugate pair in the forms:

$$z_n = w''_{n-1}(w_{n-2}w'_{n-1})$$

and

$$w_n = (w_{n-2}w'_{n-1})w''_{n-1}.$$

Since $w''_{n-1}, w_{n-2}w'_{n-1} \in X^+$ and $z_n, w_n \in Q$, by Lemma 5.1, we have that $z_n \neq w_n$.

Lemma 5.3. *Let F^1_1, F^1_2 be the codes defined in Equations (4.3), (4.4) respectively. Let $w \in F^1_{a,b}$. The following are true:*
(1) If $w \in F^1_1$ and $u \in F^1_1$ is a prefix of w, then $w = uvx$ for some $v \in F^1_2$ and $x \in X^*$.
(2) If $w \in F^1_2$ and $u \in F^1_2$ is a prefix of w, then $w = uvx$ for some $v \in F^1_1$ and $x \in X^*$.

Proof. The result is clear from the proof of Lemmas 4.4 and 4.5.

Lemma 5.4. *Let $w_n \in F^1_{a,b}$, $n \geq 1$. Then, for the Fibonacci word w_i, the following are true:* (1) $(w_i)^2 \not\leq_p w_n$, *for* $i < n$; (2) $w_i \leq_s w_n$, *for* $n \geq 3$ *and* $2 \leq i \leq n-1$; (3) $(w_i)^2 \leq_s w_n$, *for* $n \geq 6$ *and* $4 \leq i \leq n-2$; (4) $(w_i)^3 \not\leq_s w_n$, *for* $i < n$.

Proof. (1) Since the words in F^1_1 start with the letter a and the words in F^1_2 start with the letter b, the result holds from Lemmas 4.4 and 4.5.
(2) Since $w_n = w_{n-2}w_{n-1} = w_{n-2}w_{n-3}w_{n-2} = \cdots$ and by the recursion of the words in $F^1_{a,b}$, we conclude that the result is true.
(3) Consider $w_k = w_{k-2}w_{k-1}, k > 3$ and from Lemma 4.2(1), $w_{k-1}w_{k-1} = w_{k-3}w_k$, we have

$$\begin{aligned} w_k w_{k-1} &= w_{k-2}w_{k-1}w_{k-1} \\ &= w_{k-2}w_{k-3}w_k. \end{aligned} \tag{5.1}$$

By applying the Equation (5.1), we have $w_{n-2}w_{n-3} = w_{n-4}w_{n-5}w_{n-2}$ and

$$\begin{aligned} w_n &= w_{n-2}w_{n-1} \\ &= w_{n-2}w_{n-3}w_{n-2} \\ &= w_{n-4}w_{n-5}w_{n-2}w_{n-2}. \end{aligned} \tag{5.2}$$

We now show that $(w_i)^2 \leq_s w_n$, for $n \geq 6$ and $4 \leq i \leq n-2$. We do this by induction on n.
(i) For $n = 6$, $w_6 = bababbab$, since $(w_4)^2 \leq_s w_6$ and $n - 2 = 4$, the result is true.

(ii) Suppose for some $k \geq 6$, and for every $6 \leq n \leq k$, such that $(w_i)^2 \leq_s w_n, 4 \leq i \leq n-2$.

For $n = k+1$, by definition, $w_n = w_{k-1}w_k$. By assumption $(w_i)^2 \leq_s w_k, 4 \leq i \leq k-2$ and then $(w_i)^2 \leq_s w_n$. But by Equation (5.2), then

$$w_{k+1} = w_{k-3}w_{k-4}w_{k-1}w_{k-1}$$

and then $(w_{n-2})^2 = (w_{k-1})^2 \leq_s w_n$. By induction conclusion, we have that for every $n \geq 6, (w_i)^2 \leq_s w_n, 4 \leq i \leq n-2$.

(4) First, we observe that for the case $n \leq 5, i < n$ and for $n \geq 6$, along with the condition $i = n-1$, we have $(w_i)^3 \not\leq_s w_n$.

We now show that $(w_i)^3 \not\leq_s w_n$, for $n \geq 6$ and $i \leq n-2$. We do this by induction on n.

(i) For $n = 6$, $w_6 = bababbab$ and by observation $(w_1)^3, (w_2)^3, (w_3)^3$ and $(w_4)^3$ are not suffix of w_6. For $n=7$, $w_7 = abbabbababbab$. Again by observation $(w_1)^3, (w_2)^3, (w_3)^3$, $(w_4)^3$ and $(w_5)^3$ are not suffix of w_7.

(ii) Suppose for some $k \geq 6$ and for every $6 \leq n \leq k$, $(w_i)^3 \not\leq_s w_n, i \leq n-2$.

Consider $n = k+1$ and $w_n = w_{k+1} = w_{k-1}w_k$. By assumption, if $i \leq n-3 = k-2$, then $(w_i)^3 \not\leq_s w_k \leq_s w_{k+1}$. Now consider $i = n-2 = k-1$, applying the Equation (5.2), we have that

$$\begin{aligned} w_n &= w_{k+1} \\ &= w_{k-3}w_{k-4}w_{k-1}w_{k-1}. \end{aligned}$$

Since $lg(w_{k-3}w_{k-4}) = lg(w_{k-4}w_{k-3}) = lg(w_{k-2}) < lg(w_{k-1})$, then we have $(w_{k-1})^3 \not\leq_s w_{k+1}$. Thus by induction conclusion, we have $(w_i)^3 \not\leq_s w_n, i \leq n-2$.

Lemma 5.5. *Let $w_n \in F^1_{a,b}$, $n \geq 6$. Then $(w_{n-4}w_{n-5})^2 w_{n-4} \leq_p w_n$ and $(w_{n-4}w_{n-5})^3 \not\leq_p w_n$.*

Proof. Consider $w_n = w_{n-2}w_{n-1}$. Then

$$\begin{aligned} w_n &= w_{n-2}w_{n-1} \\ &= w_{n-4}w_{n-3}w_{n-3}w_{n-2} \\ &= w_{n-4}w_{n-5}w_{n-4}w_{n-5}w_{n-4}w_{n-2}. \end{aligned}$$

Thus $(w_{n-4}w_{n-5})^2 w_{n-4} \leq_p w_n$. Now by the properties of F^1_1 and F^1_2, we have $w_{n-5} \not\leq_p w_{n-2}$. It follows that $(w_{n-4}w_{n-5})^3 \not\leq_p w_n$.

Proposition 5.6. *Let $w_n \in F^1_{a,b}$, $n \geq 1$ and let $v \leq_s w_n$. If $v \leq_p w_k$, for some $w_k \in F^1_{a,b}, k \geq 1$, then $v \in F^1_{a,b}$.*

Proof. We prove the proposition by induction on the term n.

(1) By observation, the proposition is true for the first 5 terms.

(2) Assume the proposition is true for the first n terms, $n \geq 1$.

Consider $w_{n+1} = w_{n-1}w_n$. Let $v \leq_s w_{n+1}$ and $v \leq_p w_k$, for some $w_k \in F^1_{a,b}$. If $lg(v) \leq lg(w_n)$, then $v \leq_s w_n$ and, by assumption, $v \in F^1_{a,b}$. If $lg(v) > lg(w_n)$ and let $v = uw_n$, $u \in X^+$. Then $u \leq_s w_{n-1}$. From $v \leq_p w_k$, we have $u \leq_p v \leq_p w_k$. By induction hypothesis, $u \in F^1_{a,b}$. Thus $u = w_i$ for some $i \leq n-1$. We claim that $i = n-1$. Indeed, if $i \neq n-1$, then there exists an integer $j \geq 2$ such that $i = n-j$.

(2-1) If j is even, then $u = w_{n-j} \leq_p w_n$ and then $(w_{n-j})^2 \leq_p v \leq_p w_k$. This contradicts to Lemma 5.4(1).

(2-2) If j is odd, then, by Lemma 5.3, $w_{n-j+3} \leq_p w_n$. But, by Lemma 4.2(1), $w_{n-j+2}w_{n-j+2} = w_{n-j}w_{n-j+3}$ and $(w_{n-j+2})^2 = w_{n-j}w_{n-j+3} = uw_{n-j+3} \leq_p v \leq_p w_k \in F^1_{a,b}$

Again this contradicts to Lemma 5.4(1). We then have $v = uw_n = w_{n-1}w_n = w_{n+1} \in F^1_{a,b}$. This completes our induction proof and the proposition holds.

Proposition 5.7. *Let w_n be the n-th term of the Fibonacci language $F^1_{a,b}$, $n \geq 1$ and let*

$$w_{n+2} = w_n w_{n+1} = x(w_i)^3 y, \tag{5.3}$$

where $w_i \in F^1_{a,b}, i \geq 1$ and $x \leq_p w_n$, $y \leq_s w_{n+1}$. Suppose w_i satisfies Equation (5.3) with $lg(w_i) \geq lg(w_j)$, for every j satisfies Equation (5.3). Then the following are true:

(1) *If $w_i \leq_p w_{n+1}$, then $i = n-3$. That is, $w_{n-3} \leq_p w_{n+1}$ and $(w_{n-3})^2 \leq_s w_n$.*

(2) *If $w_i \not\leq_p w_{n+1}$, then $i = n-2$. That is, $w_{n-4} \leq_s w_n$ and $w_{n-3}(w_{n-2})^2 \leq_p w_{n+1}$.*

Proof. (1) By observation, for $n < 6$, we have that $w_{n+2} \notin G_3$. Consider $w_{n+1} = w_{n-1}w_n$, for $n \geq 6$. Suppose $w_i \leq_p w_{n+1}$. Then by the properties of words in $F^1_{a,b}$, $w_{n-k} \leq_p w_{n+1}$; for $k \geq 1$ and k is odd. From the fact that $lg((w_{n-1})^2) > lg(w_n)$, then $(w_{n-1})^2 \not\leq_s w_n$ holds. Now by Lemma 5.4 and the fact that w_i is of maximal length in the sense that $w_{n+2} = w_n w_{n+1} = x(w_i)^3 y$, we must have $i = n-3$ such that $(w_i)^2 \leq_s w_n, w_i \leq_p w_{n+1}$ and $(w_i)^3 \in E(w_{n+2})$.

(2) Suppose $w_i \not\leq_p w_{n+1}$. Then let $(w_i)^r u_1 \leq_s w_n$, $u_2(w_i)^s \leq_p w_{n+1}$, where $r, s \geq 0, r+s = 2$, $w_i = u_1u_2$, $u_1, u_2 \in X^+$. It is clear that $i \leq n-1$. But $w_{n-1} \leq_p w_{n+1}$ and hence $i \leq n-2$. By Lemma 5.4(3), we have $(w_i)^2 \leq_s w_n$.

(i) If $r \geq 1$, then $(w_i)^r u_1 \leq_s w_n$. Thus since $(w_i)^2 \leq_s w_n$ and $w_i = u_1u_2$, the condition $u_2u_1 = u_1u_2$ holds. By Lemma 1.7([4]), we have that u_1 and u_2 are powers of a common word and then $w_i \notin Q$, a contradiction.

(ii) If $r = 0$, then $u_1 \leq_s w_n$ and $u_2(w_i)^2 \leq_p w_{n+1}$. Since $u_1 \leq_p w_i$ and $u_2 \leq_s w_i$, by Proposition 5.6, both u_1 and u_2 are in $F^1_{a,b}$. It follows that $u_1 = w_{i-2}, u_2 = w_{i-1}$. That is $w_{i-1}(w_i)^2 \leq_p w_{n+1}$. Consider

$$\begin{aligned} w_{n+1} &= w_{n-1}w_n \\ &= w_{n-3}w_{n-2}w_{n-2}w_{n-1} \end{aligned}$$

and by Lemma 5.4(2), $w_{n-4} \leq_s w_n$. Hence $i = n-2$ and $(w_{n-2})^3 \in E(w_{n+2})$.

By the above proposition, we see that $F^1_{a,b} \cap G_3 \neq \emptyset$ and it is also true that $F^0_{a,b} \cap G_3 \neq \emptyset$. We now give some concrete examples as follows.

In $F^0_{a,b}$, $z_1 = a$, $z_2 = b$, $z_3 = ba$, $z_4 = bab$, $z_5 = babba$, $z_6 = babbabab$, $z_7 = babbababbabba$, $z_8 = babba(bab)(bab)(bab)abbabab$ and $(bab)^3$ is a subword of z_8.
In $F^1_{a,b}$, $w_1 = a$, $w_2 = b$, $w_3 = ab$, $w_4 = bab$, $w_5 = abbab$, $w_6 = bababbab$, $w_7 = abbabbababbab$, $w_8 = bababba(bab)(bab)(bab)abbab$, and $(bab)^3$ is a subword of w_8.
There is a word $u \in X^+, u \notin F^1_{a,b}$ such that $u^3 \in E(w_n)$, $w_n \in F^1_{a,b}$. This can be seen from the term,

$$\begin{aligned} w_9 &= abbabbababb(abbab)(abbab)(abbab)bababbab \\ &= abbabbabab(babba)(babba)(babba)bbababbab. \end{aligned}$$

Where both $abbababbababbab = (ababb)^3$ and $babbababbababba = (babba)^3$ are subwords of w_9.

In the final part of this paper we will show that for any $u \in X^+$, u^4 is not a subword of any Fibonacci word.

Proposition 5.8. *Let $w_n \in F^1_{a,b}$, $n \geq 1$. Then $(w_i)^4 \notin E(w_n)$, for every word $w_i \in F^1_{a,b}, i < n$.*

Proof. By observation, the proposition holds true for $n \leq 9$. We now show that the proposition is true for all $n \geq 10$. Suppose, for $n \geq 8$,

$$w_{n+2} = w_n w_{n+1} = x(w_i)^4 y, \text{ for some } i.$$

Then by Proposition 5.7, i can not be $n-1$. Consider $i = n-2$ and $i = n-3$. By Proposition 5.7, we have that one of the following two conditions holds:
(1) If $w_i \leq_p w_{n+1}$, then $i = n-3$, i.e., $w_{n-3} \leq_p w_{n+1}$ and $(w_{n-3})^2 \leq_s w_n$. If (1) holds, then by Lemma 5.4(1), $(w_{n-3})^2 \not\leq_p w_{n+1}$ and by Lemma 5.4(4), $(w_{n-3})^3 \not\leq_s w_n$, we have that $(w_{n-3})^4 \notin E(w_{n+2})$.
(2) If $w_i \not\leq_p w_{n+1}$, then $i = n-2$, i.e., $w_{n-4} \leq_s w_n$ and $w_{n-3}(w_{n-2})^2 \leq_p w_{n+1}$.
If (2) holds, then (i) we claim that $w_{n-2}w_{n-4} \not\leq_s w_n$. If not, then $w_{n-2}w_{n-4} = w_{n-4}w_{n-3}$ w_{n-4} and we have $w_{n-3}w_{n-4} \leq_s w_n$. But by Lemma 5.4(2), $w_{n-2} \leq_s w_n$. Since $lg(w_{n-2}) = lg(w_{n-4}w_{n-3}) = lg(w_{n-3}w_{n-4})$, we have that $w_{n-4}w_{n-3} = w_{n-3}w_{n-4}$. By Lemma 1.7([3]), w_{n-3} and w_{n-4} are powers of a common word and then $w_{n-2} \notin Q$, a contradiction. So $w_{n-2}w_{n-4} \not\leq_s w_n$. (ii) We claim that $w_{n-3}(w_{n-2})^3 \not\leq_p w_{n+1}$. By Lemma 5.5, $(w_{n-3}w_{n-4})^3 = w_{n-3}(w_{n-4}w_{n-3})^2 w_{n-4} \not\leq_p w_{n+1}$ and then $w_{n-3}(w_{n-4}w_{n-3})^3 \not\leq_p w_{n+1}$. Thus we have that $w_{n-3}(w_{n-2})^3 = w_{n-3}(w_{n-4}w_{n-3})^3 \not\leq_p w_{n+1}$. By (i) and (ii), we have that $(w_{n-2})^4 \notin E(w_{n+2})$.
Finally, for $i < n-3$, by the recursion of the words in $F^1_{a,b}$, we conclude that $(w_i)^4 \notin E(w_{n+2})$ and the proof is completed.

If w_n is the n-th term of $F^1_{a,b}$, $n \geq 4$, then w_n is an overlapping word. This can be seen from the fact that

$$w_n = w_{n-2}w_{n-3}w_{n-2}.$$

Proposition 5.9. *Let $w_n \in F^1_{a,b}$, $n \geq 1$. For $w_n \neq u \in X^+$, if $u \leq_p w_n$ and $u \leq_s w_n$, then $u \in \{w_i | n-2 \geq i \geq 2, n-i$ is even $\}$.*

Proof. Let $u \in X^+$. If $u \leq_p w_n$ and $u \leq_s w_n$, then, by Proposition 5.6 and Lemma 5.3, we have $u \in \{w_i | n-2 \geq i \geq 2, n-i \text{ is even}\}$.

Lemma 5.10. *Let $w_n \in F^1_{a,b}$, $n \geq 1$. Then the following two conditions hold:*
(1) *$w_n \not\leq_p w_{n+1}$ and $w_n \notin \bar{E}(w_{n+1})$,*
(2) *for $n \geq 2$, $w_n \notin \bar{E}(w_n w_{n-1})$.*

Proof. (1) By observation, we see that $w_n \not\leq_p w_{n+1}$. We now show that $w_n \notin \bar{E}(w_{n+1})$. Suppose $w_n \in \bar{E}(w_{n+1})$. Then, for some $x, y \in X^+$,

$$w_{n+1} = w_{n-1}w_n = xw_ny. \tag{5.4}$$

Thus $lg(x) + lg(y) = lg(w_{n-1})$ and there exist $w'_n, w''_n \in X^+$ such that

$$w_n = w''_n y = w'_n w''_n.$$

Then, by Proposition 5.9, $w''_n \in \{w_i | n-2 \geq i \geq 2, n-i \text{ is even}\}$. Now let $w''_n = w_i$, for some $n-2 \geq i \geq 2$. Consider $w_n = w_{n-2}w_{n-1} = w_{n-2}w_{n-3}w_{n-2}$. If $lg(w_i) = lg(w''_n) = lg(w_{n-2})$, then $lg(y) = lg(w_{n-1})$. From Equation (5.4), we see that $lg(x) = 0$, a contradiction. Therefore $lg(w_i) < lg(w_{n-2})$. From $w_n = w''_n y = w_i y$, we must have $lg(y) > lg(w_{n-1})$. This contradicts to that $lg(x) + lg(y) = lg(w_{n-1})$. The statement (1) is then true. (2) By similar argument as in the proof of (1), we can prove (2).

Lemma 5.11. *Let $w_n \in F^1_{a,b}$, $n \geq 1$. If for some $u \in X^+$, $u^2 \in E(w_n)$, then $lg(u) \leq lg(w_{n-2})$. In other words, if $lg(u) > lg(w_{n-2})$, then $u^2 \notin E(w_n)$.*

Proof. Suppose $u^2 \in E(w_n)$ for some $n \geq 1$ and $lg(u) > lg(w_{n-2})$. Then there exist $x, y, u_1, u_2, u_3, u_4 \in X^*$ such that

$$w_n = w_{n-2}w_{n-3}w_{n-2} = xu_1u_2u_3u_4y,$$

where $u = u_1u_2 = u_3u_4$; $w_{n-3} = u_2u_3$; $w_{n-2} = xu_1 = u_4y$. Now
(1) if $lg(u_1) \leq lg(u_3)$, then

$$lg(u) = lg(u_1) + lg(u_2) \leq lg(u_3) + lg(u_2) = lg(w_{n-3}) < lg(w_{n-2}),$$

which is not true.
(2) If $lg(u_1) > lg(u_3)$, then from the condition that

$$lg(u) = lg(u_3) + lg(u_4) > lg(w_{n-2}) = lg(u_4) + lg(y),$$

we see that $lg(u_3) > lg(y)$ and $u_3 \in X^+$. Since $lg(u_1) > lg(u_3)$, there exists $z \in X^+$ such that

$$u_1 = u_3z, \quad \text{and} \quad u_4 = zu_2.$$

And then

$$w_{n-2} = xu_3z = zu_2y.$$

Since $z \leq_p w_{n-2}$, $z \leq_s w_{n-2}$ and $u_3 \in X^+$, then $z \neq w_{n-2}$ and then by Proposition 5.9, $lg(z) \leq lg(w_{n-4})$. Now from $lg(u_3) > lg(y)$, we have $lg(u_2y) < lg(u_2u_3) = lg(w_{n-3})$ and then $lg(z) > lg(w_{n-4})$, a contradiction.
By (1) and (2), the result is true.

Lemma 5.12. *Let* $w_n \in F^1_{a,b}$ $n \geq 1$. *If* $lg(u) = lg(w_{n+1})$, *then* $u^2 \notin E(w_nw_{n+1}w_n)$. *Moreover, if* $u^2 \in E(w_{n+1}w_{n+1}w_n)$, *then* $u = w_{n+1}$. *That is,* $u^2 \notin \bar{E}(w_{n+1}w_{n+1}w_n)$.

Proof. Suppose $lg(u) = lg(w_{n+1})$ and $u^2 \in E(w_nw_{n+1}w_n)$. Let

$$w_nw_{n+1}w_n = xu_1u_2u_3u_4y,$$

where $x, y, u_1, u_2, u_3, u_4 \in X^*$; $u = u_1u_2 = u_3u_4$ and $w_n = xu_1 = u_4y$; $w_{n+1} = u_2u_3$. Since $lg(u) = lg(w_{n+1})$, we have $lg(u_1) = lg(u_3), lg(u_2) = lg(u_4)$. By $u_1 \leq_s w_n$, $u_3 \leq_s w_{n+1}$, then $u_1 = u_3$. But $u_2 \leq_p w_{n+1}$, $u_4 \leq_p w_n$, by properties of the sets F^1_1 and F^1_2, we see that $u_2 \neq u_4$. It follows that $u = u_1u_2 \neq u_3u_4 = u$, a contradiction. The former part of the lemma is then true.

By the similar argument as in the proof of first part, we are able to show that $u^2 \notin E(w_{n+1}w_{n+1}w_n)$.

Lemma 5.13. *Let* $w_n \in F^1_{a,b}$, $n \geq 1$. *If* $lg(w_{n+1}) > lg(u) > lg(w_n)$, *then* (1) $u^3 \notin E(w_{n+1}w_nw_{n+1}w_{n+1})$; (2) $u^3 \notin E(w_{n+1}w_nw_{n+1}w_n)$.

Proof. (1) Suppose $u^3 \in E(w_{n+1}w_nw_{n+1}w_{n+1})$ and for $x, y, u_1, u_2, u_3, u_4, u_5, u_6 \in X^*$,

$$w_{n+1}w_nw_{n+1}w_{n+1} = xu_1u_2u_3u_4u_5u_6y,$$

where $u = u_1u_2 = u_3u_4 = u_5u_6$; $w_{n+1} = xu_1 = u_4u_5 = u_6y$; $w_n = u_2u_3$. Since $lg(w_{n+1}) > lg(u) > lg(w_n)$, by assumption, we have $lg(u_1) > lg(u_3), lg(u_3) < lg(u_5)$. Thus there exist $z_1, z_2 \in X^+$ such that $u_1 = u_3z_1$, $u_4 = z_1u_2$; $u_5 = u_3z_2$, $u_4 = z_2u_6$. Hence

$$w_{n+1} = u_4u_5 = z_1u_2u_3z_2 = z_1w_nz_2.$$

This contradicts to Lemma 5.10(1) and condition of Lemma 5.13 (1) holds true.
(2) Similar argument as the proof of (1) will work for the proof of (2).

By the same technique used in the proof of Lemma 5.13, we are able to show the following:

Corollary 5.14 Let $w_n \in F^1_{a,b}$, $n \geq 1$. If $lg(w_{n+1}) > lg(u) > lg(w_n)$, then (1) $u^3 \notin E(w_{n+1}w_{n+1}w_nw_{n+1})$; (2) $u^3 \notin E(w_nw_{n+1}w_nw_{n+1})$.

Proposition 5.15. *Let* $w_n \in F^1_{a,b}, n \geq 1$. *Then* $w_n \notin G_4$.

Proof. By observation the proposition is true for $n \leq 7$. Now suppose there exists a $j \geq 1$ such that $w_j \in G_4$. Let w_m be the first Fibonacci word in $F^1_{a,b} \cap G_4$. Then $w_{m-1}, w_{m-2} \notin G_4$. By Proposition 5.8, we have that, $(w_i)^4 \notin E(w_m)$, for $w_i \in F^1_{a,b}$, $1 \leq i < m$. Thus we consider for the case $u \notin F^1_{a,b}$ and $u^4 \in E(w_m)$. Let

$$w_m = w_{m-2}w_{m-1} = xu^4y, \text{ for some } x, y \in X^*.$$

Then since, by assumption, w_m is the first Fibonacci word in $F^1_{a,b} \cap G_4$, we have $x \leq_p w_{m-2}$, $y \leq_s w_{m-1}$.

(A) If $lg(u) > lg(w_{m-3})$, then by Lemma 5.11, $u^2 \notin E(w_{m-1})$ and then $u^4 \notin E(w_m)$.

(B) If $lg(u) \leq lg(w_{m-6})$, then, by $lg(w_n) = lg(w_{n-2}) + lg(w_{n-1})$ and $3 \cdot lg(w_{n-1}) < 2 \cdot lg(w_n)$,

$$\begin{aligned} lg(u^4) &\leq 4 \cdot lg(w_{m-6}) \\ &= lg(w_{m-6}) + 3 \cdot lg(w_{m-6}) \\ &< lg(w_{m-6}) + 2 \cdot lg(w_{m-5}) \\ &= lg(w_{m-4}) + lg(w_{m-5}) \\ &= lg(w_{m-3}). \end{aligned}$$

Thus $lg(u^4) < lg(w_{m-3})$. Since $w_m = w_{m-2}w_{m-1} = w_{m-4}w_{m-3}w_{m-3}w_{m-2}$ and $u^4 \notin E(w_{m-2}), u^4 \notin E(w_{m-1})$, we have $u^4 \in E(w_{m-3}w_{m-3})$. But $w_{m-3}w_{m-3} \leq_s w_{m-1}$ (see Equation (5.2)), we have that $u^4 \in E(w_{m-1})$ and this contradict to that w_m is, by assumption, the first Fibonacci word such that $u^4 \in E(w_m)$.

By above (A) and (B), we have if $u^4 \in E(w_m)$, then

$$lg(w_{m-6}) < lg(u) \leq lg(w_{m-3}).$$

In order to complete the proof of that no $u \in X^+ \setminus F^1_{a,b}$ such that $u^4 \in E(w_n)$, for $w_n \in F^1_{a,b}$, we consider the following three equations :

$$w_m = w_{m-2}w_{m-3}w_{m-2} = w_{m-4}w_{m-3}w_{m-3}w_{m-4}w_{m-3} \tag{5.5}$$

$$\begin{aligned} w_m &= w_{m-4}w_{m-5}w_{m-4}w_{m-5}w_{m-4}w_{m-4}w_{m-5}w_{m-4} \\ &= w_{m-6}w_{m-7}w_{m-4}w_{m-4}w_{m-5}w_{m-4}w_{m-4}w_{m-5}w_{m-4} \end{aligned} \tag{5.6}$$

$$w_m = w_{m-6}w_{m-5}w_{m-5}w_{m-6}w_{m-5}w_{m-5}w_{m-6}w_{m-5}w_{m-4}w_{m-3} \tag{5.7}$$

(1) Consider $lg(u) = lg(w_i), i = m-3, m-4, m-5$. In each case, by Equations (5.5), (5.6) and (5.7) and looking at the length of u, we see that $u^2 \in E(w_iw_iw_{i-1})$. But by Lemma 5.12 $u = w_i$, this contradicts to $u \notin F^1_{a,b}$. Thus we have $u^2 \notin E(w_iw_iw_{i-1})$. Hence $u^4 \notin E(w_m)$.

We now consider the remaining cases:

(2) If $lg(w_{m-4}) < lg(u) < lg(w_{m-3})$, then by Equation (5.5) we have that $u^3 \in E(w_{m-3}w_{m-3}w_{m-4}w_{m-3})$. It contradicts to Corollary 5.14(1).

(3) If $lg(w_{m-5}) < lg(u) < lg(w_{m-4})$, then by Equation (5.6), we have that either
(3-1) $u^3 \in E(w_{m-4}w_{m-5}w_{m-4}w_{m-5})$, This contradicts to Lemma 5.13(2); or
(3-2) $u^3 \in E(w_{m-4}w_{m-5}w_{m-4}w_{m-4})$, this contradicts to Lemma 5.13(1).
(4) If $lg(w_{m-6}) < lg(u) < lg(w_{m-5})$, then by Equation (5.7), we have that either
(4-1) $u^3 \in E(w_{m-5}w_{m-6}w_{m-5}w_{m-5})$, this contradicts to Lemma 5.13(1); or
(4-2) $u^3 \in E(w_{m-5}w_{m-5}w_{m-6}w_{m-5})$, this contradicts to Corollary 5.14(1).
By above (1), (2), (3) and (4), we complete the proof of that there is no word $u \in X^+ \backslash F^1_{a,b}$ with $lg(w_{m-6}) < lg(u) \leq lg(w_{m-3})$ such that $u^4 \in E(w_m)$.
The proof of the proposition is then completed.

For $u, v \in X^+$, we call v the *mirror image* of u if $u = a_1a_2\cdots a_r$, for some $r \geq 1$ and $a_1, a_2, \ldots, a_r \in X$, then $v = a_ra_{r-1}\cdots a_1$. Now we can prove the following corollary.

Corollary 5.16. *Let $z_n \in F^0_{a,b}$ $n \geq 1$. Then $z_n \notin G_4$.*

Proof. If w_n is the n-th term of $F^1_{a,b}$ and z_n is the n-th term of $F^0_{a,b}$, then z_n is the mirror image of w_n. (This can be proved by induction on n and from the fact that $w_n = w_{n-2}w_{n-3}w_{n-2}$ and $z_n = z_{n-2}z_{n-3}z_{n-2}$.) With this fact it is clear that $z_n \notin G_4$, otherwise w_n will be in G_4.

The following is now clear:

Proposition 5.17. *The atom Fibonacci languages $F^0_{a,b}$ and $F^1_{a,b}$ are subsets of $G^{(1)} \cup G^{(2)} \cup G^{(3)}$.*

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