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RINGS WITH (R, R, R) AND $[R, (R, R, R)]$ IN THE LEFT NUCLEUS

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Abstract. Let R be a nonassociative ring, and $N = (R, R, R) + [R, (R, R, R)]$. We show that $W = \{\omega \varepsilon N \mid R\omega + \omega R + R(\omega R) \subset N\}$ is a two-sided ideal of R. If for some $r \varepsilon R$, any one of the sets (r, R, R), (R, r, R) or (R, R, r) is contained in W, then the other two sets are contained in W also. If the associators are assumed to be contained in either the left, the middle, or the right nucleus, and I is the ideal generated by all associators, then $I^2 \subset W$. If N is assumed to be contained in the left or the right nucleus, then $W^2 = 0$. We conclude that if R is semiprime and N is contained in the left or the right nucleus, then R is associative. We assume characteristic not 2.

I. Introduction

In this paper R is assumed to be a nonassociative ring. The associator (a, b, c) and commutator $[a, b]$ are defined by $(a, b, c) = (ab)c - a(bc)$, $[a, b] = ab - ba$. The left nucleus $= \{r \mid (r, R, R) = 0\}$. The middle nucleus $= \{r \mid (R, r, R) = 0\}$. The right nucleus $= \{r \mid (R, R, r) = 0\}$. The paper starts by considering the general nonassociative ring in Section II. In Sections III and IV it adds additional assumptions which study:
III. When (R, R, R) is contained in the left, the middle, or the right nucleus.
IV. When $(R, R, R) + [R, (R, R, R)]$ is contained in the left nucleus and R is semiprime.

The literature contains related work by Kleinfeld, and Yen. Kleinfeld [1] studied the case where associators were assumed to be simultaneously contained in the left, middle, and right nucleus. He proved that semiprime rings were associative. Yen [2] assumed associators simultaneously contained in the left and middle nucleus. He showed that simple rings were associative. Yen [3] improved his previous paper by assuming that (R, R, R) and $[R, (R, R, R)]$ are contained in the left nucleus. This is a weakening of his original assumptions. He proved that simple rings were associative.

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This current paper improves on [3] by replacing simple by semiprime. This paper also introduces an ideal W of the general nonassociative algebra. If I is the ideal generated by all associators, then $W \subset I$. If the associators are assumed to be contained in any one of the three nuclei, then $I^2 \subset W$. Under certain additional assumptions we get $W^2 = 0$. This sets the stage for semiprime to imply associativity.

Throughout the paper we reserve N to be $N = (R,R,R) + [R,(R,R,R)]$ and $W = \{\omega \varepsilon N \mid R\omega + \omega R + R(\omega R) \subset N\}$. I will be the ideal generated by all associators. Multiplication is indicated both by juxtaposition and by "$\times$". When both forms occur, juxtaposition takes precedence. So $(ab \times c) \times de$ means $((ab)c)(de)$.

II. The ideal W in any nonassociative ring R

We will start by proving some properties about an ideal contained in $N = (R,R,R)+ [R,(R,R,R)]$. The following is the Teichmüller identity which holds in any nonassociative ring.

$$(ab,c,d) - (a,bc,d) + (a,b,cd) = a(b,c,d) + (a,b,c)d. \tag{1}$$

Lemma 1. *Let r be any element in R. Then:*

a) $R(r,R,R) + R(R,r,R) + R(R,R,r) \subset r(R,R,R) + N.$

b) $(r,R,R)R + (R,r,R)R + (R,R,r)R \subset r(R,R,R) + N.$

Proof of (a). We add together Eq. (1) and $[d,(a,b,c)] = d(a,b,c) - (a,b,c)d$ to get: $(ab,c,d) - (a,bc,d) + (a,b,cd) + [d,(a,b,c)] = a(b,c,d) + d(a,b,c)$. We conclude that modulo N, we can perform cyclic shifts on the arguments of $a(b,c,d)$ until any of the arguments appears in the first position. The proof of (b) follows from the proof of (a) because $[R,(R,R,R)] \subset N$.

We now examine an ideal which is contained in N in any nonassociative algebra R.

Theorem 2. *Let $W = \{\omega \varepsilon N \mid R\omega + \omega R + R(\omega R) \subset N\}$. Then W is a two-sided ideal of R.*

Proof. We will first show that W is a right ideal by showing that for any ω in W and r in R we get that wr is in W. There are four items to check.

a) $\omega r \varepsilon N$ by defintion of W.

b) $R(\omega r) \subset R \times \omega R \subset N$ by definition of W.

c) $(\omega r)R \subset (\omega, r, R) + \omega \times rR \subset (R,R,R) + \omega R \subset N$ by definitions of W and N.

d) $R \times (\omega r \times R) \subset R \times (\omega, r, R) + R(\omega \times rR) \subset \omega(R,R,R) + N + R \times \omega R \subset N$ by Lemma 1(a) and the definition of W.

We will now show that W is a left ideal by showing that ω in W and r in R implies $r\omega$ is in W.

e) $r\omega \varepsilon N$ by definitions of W.

f) $R(r\omega) \subset (R,r,\omega) + Rr \times \omega \subset (R,R,R) + R\omega \subset N$ by definitions of W and N.

g) $(r\omega)R \subset (r,\omega,R) + r \times \omega R \subset (R,R,R) + R \times \omega R \subset N$ by definitions of W and N.

h)

$$\begin{aligned} R \times (r\omega \times R) &\subset R(r,\omega,R) + R \times (r \times \omega R) \\ &\subset R(r,\omega,R) + (R,r,\omega R) + Rr \times \omega R \\ &\subset \omega(R,R,R) + N + (R,R,R) + R \times \omega R \\ &\subset N \quad \text{by Lemma 1(a) and the definitions of } W \text{ and } N. \end{aligned}$$

Theorem 3. *If r is an element of R and one of the sets $(r,R,R),(R,r,R)$, or (R,R,r) is contained in W, then each of the sets is contained in W.*

Proof. We have to prove four conditions to show that $(r,R,R) + (R,r,R) + (R,R,r) \subset W$.

a) $(r,R,R) + (R,r,R) + (R,R,r) \subset (R,R,R) \subset N$ by definition of N.

b) By the proof of Lemma 1(a), $a(b,c,d) \equiv -d(a,b,c)$ modulo N. We conclude that modulo N, we can cycle the arguments of $a(b,c,d)$. This means that if any one of $(r,R,R),(R,r,R)$, or (R,R,r) is contained in W, then $R(r,R,R) + R(R,r,R) + R(R,R,r) \subset RW + N \subset N$. We have used the fact that W is an ideal and $W \subset N$.

c) Using Part b) we get $(r,R,R)R + (R,r,R)R + (R,R,r)R \subset N$ whenever any one of $(r,R,R),(R,r,R)$ or (R,R,r) is contained in W because $[R,(R,R,R)] \subset N$.

d) To prove the fourth part, we will show how the entries of the general expression $a \times (b,c,d)e$ can be shifted modulo N.

$$\begin{aligned} \text{(d.1)} \quad a \times (b,c,d)e &= [a,(b,c,d),e] + (b,c,d)e \times a \\ &\subset [a,-b(c,d,e) + (R,R,R)] + (b,c,d) \times ea + (R,R,R) \quad \text{by Eq.(1).} \\ &\subset N - [a,b(c,d,e)] + (b,c,d) \times ea \quad \text{by definition of } N. \\ &\subset N + [R,R(c,d,R)] + (b,c,d)R \end{aligned}$$

$$\begin{aligned} \text{(d.2)} \quad a \times (b,c,d)e &= -(a,(b,c,d),e) + a(b,c,d) \times e \\ &\subset (R,R,R) + [a(b,c,d),e] + e \times a(b,c,d) \\ &\subset (R,R,R) + [(R,R,R) - (a,b,c)d,e] + ea \times (b,c,d) + (R,R,R) \\ &\qquad \text{by Eq.(1)} \\ &\subset N - [(a,b,c)d,e] + ea \times (b,c,d) \quad \text{by definiton of } N. \\ &\subset N + [(R,b,c)R,R] + R(b,c,d). \end{aligned}$$

Now if b,c or d is replaced by r where at least one of $(r,R,R),(R,r,R)$ or (R,R,r) is contained in W, then $R(b,c,d)$ and $(b,c,d)R$ are in N by Parts (b) and (c).

Part (d.1) with $c = r$ tells us that $(r, R, R) \subset W \Rightarrow R\times(R, r, R)R \subset W$. So $(R, r, R) \subset W$.
Part (d.1) with $d = r$ tells us that $(R, r, R) \subset W \Rightarrow R\times(R, R, r)R \subset W$. So $(R, R, r) \subset W$.
Part (d.2) with $c = r$ tells us that $(R, R, r) \subset W \Rightarrow R\times(R, r, R)R \subset W$. So $(R, r, R) \subset W$.
Part (d.2) with $b = r$ tells us that $(R, r, R) \subset W \Rightarrow R\times(r, R, R)R \subset W$. So $(r, R, R) \subset W$.
In any case, if one of (r, R, R), (R, r, R), or (R, R, r) is contained in W, then each of (r, R, R), (R, r, R) and (R, R, r) is contained in W.

III. When $((R, R, R), R, R) + (R, (R, R, R), R) + (R, R, (R, R, R)) \subset W$

In this section we will assume that $((R, R, R), R, R) + (R, (R, R, R), R) +(R, R, (R, R, R)) \subset W$. Notice that if r is in the left nucleus, the middle nucleus, or the right nucleus, then at least one of $(r, R, R), (R, r, R)$ or (R, R, r) is zero and is contained in W. By Theorem 3 we conclude three sets are contained in W. This section is a generalization of the assumption that the associators are in the left nucleus, the middle nucleus, or the right nucleus.

Theorem 4. (Kleinfeld) *If* $((R, R, R), R, R) + (R, (R, R, R), R)+ (R, R, (R, R, R)) \subset W$, *then* $2(R, R, R)(R, R, R) \subset W$.

Proof. Eq. (1) has 4 arguments and 5 terms. If we replace any one of the arguments with an associator, then 3 of the five terms involve types of the form $((R, R, R), R, R)$, $(R, (R, R, R), R)$ or $(R, R, (R, R, R))$ and are contained in W by assumption and the fact that W is an ideal. Working modulo W and using Eq. (1) repeatedly, we get the following:

$$\begin{aligned}(a, b, c)(d, e, f) &\equiv ((a, b, c)d, e, f) \equiv -(a(b, c, d), e, f) \equiv -(a, (b, c, d)e, f)\\ &\equiv +(a, b(c, d, e), f) \equiv +(a, b, (c, d, e)f) \equiv -(a, b, c(d, e, f)) \equiv -(a, b, c)(d, e, f).\end{aligned}$$

We conclude that $2(R, R, R)(R, R, R) \subset W$.

We use I to be the ideal generated by all associators. It is an easy task using Eq. (1) to show that in any nonassociative ring, $I = (R, R, R) + (R, R, R)R = (R, R, R) + R(R, R, R)$.

Theorem 5. (Kleinfeld) *If* $((R, R, R), R, R)+(R, (R, R, R), R)+(R, R, (R, R, R)) \subset W$, *then* $2I^2 \subset W$.

Proof. $I^2 = \{(R, R, R) + (R, R, R)R\}I \subset (R, R, R)I + (R, R, R) \times RI +((R, R, R), R, I) \subset (R, R, R)I + W \subset (R, R, R)\{(R, R, R) + (R, R, R)R\} + W \subset (R, R, R)^2 + (R, R, R)^2R + W$. We conclude that $2I^2 \subset W$ from Theorem 4.

IV. $W \subset$ the left nucleus, R is semiprime

We assume that W is contained in the left nucleus. We get the same results if we assume that W is contained in the right nucleus. Our assumption is weaker than assuming that $N \subset$ the left nucleus.

Theorem 6. *If $W \subset$ the left nucleus, then $W^2 = 0$*

Proof. $W^2 \subset WI \subset W\{(R,R,R)+(R,R,R)R\} \subset (WR,R,R)+(WR,R,R)R = 0$ using, the assumption that W is contained in the left nucleus, W is an ideal, and $W \subset I$.

Theorem 7. *Let R be a semiprime ring of characteristic not 2 with (R, R, R) and $[R, (R, R, R)]$ contained in the left nucleus. Then R is associative.*

Proof. Since $W \subset (R, R, R) + [R, (R, R, R)]$, by Theorem 6, $W^2 = 0$. By the assumption of semiprime, $W = 0$. Now by Theorem 3 and Theorem 5, $(2I)^2 = 4I^2 \subset W = 0$. By the assumption of semiprime, we get $2I = 0$. By characteristic not two we have $I = 0$, and R must be associative.

Theorem 8. *Let R be a ring of characteristic not 2 with (R, R, R) and $[R, (R, R, R)]$ contained in the left nucleus. Then $(R, R, R)^3 = 0$.*

Proof. Note that this result holds even if the ring is not semiprime. From Theorem 3 and Theorem 4 we have $2(R, R, R)(R, R, R) \times (R, R, R) \subset W(R, R, R) = 0$ because W is contained in the left nucleus and W is an ideal. By characteristic not 2 we get that $(R, R, R)(R, R, R) \times (R, R, R) = 0$.

V. Remark

If N is contained in the middle nucleus, then W is an associative ring. We know that W^2 is an ideal but we cannot show that it is zero. Each power of W is also an ideal. If any one of those powers were zero, then semiprime would imply that W is zero.

References

[1] E. Kleinfeld, "A class of rings which are very nearly associative," *Amer. Math. Monthly*, 93(1986), 720-722.

[2] Chen-Te Yen, "Rings with associators in the left and middle nucleus," *Tamkang J. Math.*, 23(1992), 363-369.

[3] ——, "Rings with (R, R, R) and $((R, R, R), R)$ in the left nucleus," *Tamkang J. Math.*, 24(1993), 209-213.

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