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APPLICATION OF CERTAIN FRACTIONAL CALCULUS OPERATORS IN STATISTICAL DISTRIBUTIONS

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Abstract. This paper deals with the application of certain classes of fractional calculus operators in statistical distributions. The images of product combinations of special functions under the calculus of operators are applied to certain generalized forms of univariate and multivariate statistical distributions. Further results giving the expectations, cummulative functions and characteristic functions of such special function distributions are also obtained.

1. Introduction and Prelimanaries

The theory of fractional calculus is receiving increasing attention from many researchers. After the celebrated conference at the university of New Haven (USA) in 1974 [9], several papers on different aspects relating fractional calculus and their applications have appeared including the two international conferences at the University of Strathclyde (U.K.) in 1984 and at Nihon University (Japan) in 1989. Three recent books on the subject by Miller and Ross.[8], Kiryakova [6] and Samko, Kilbas and Marichev [13] give fairly good account of the development in fractional calculus and their applications to various problems of analysis. Applications of fractional calculus operators in problems of statistics have been considered in [9], [12] and [13].

The present paper is intended to apply fractional calculus operators to certain univariate and multivariate statistical distributions. We first state few formulas giving the images under fractional calculus operators of the product combinations of elementary functions and hypergeometrie functions which are expressed in terms of generalised Kampe' de Féreit functions [15]. These results are then applied to certain classes of statistical distributions. Further results concerning the problems of finding expectations.

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Cummulative functions and characterstic functions of our special function distributions are also obtained.

Let $\alpha, \beta, \eta \in C$ with $Re\,(\alpha) > 0$. The fractional integral of the first kind of a suitable function $f(x)$ is defined by (see[10]):

$$I_{o,x}^{\alpha,\beta,\eta} f(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_{o}^{x} (x-t)^{\alpha-1} F(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}) f(t) dt, \tag{1.1}$$

for $x \in R_+$.

Here and elsewhere, C is the complex number field, $R_+ = (0, \infty)$, N_0 denotes the set of nonnegative integers. The fractional derivative of $f(x)$ for $Re(\alpha) < 0$ is defined by

$$I_{o,x}^{\alpha,\beta,\eta} f(x) = \frac{d^n}{dx^n} I_{o,x}^{\alpha+n,\beta-n,\eta-n} f(x) \tag{1.2}$$

provided that $o < Re(\alpha) + n \leq 1$, $n \in N_0$.

The Gauss hypergeometric function $F(a, b; c; z)$ appearing in (1.1) is a special case of the generalised hypergeomertric function

$$_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{m=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_m}{\prod_{j=1}^{q} (b_j)_m} \cdot \frac{z^m}{m!} \tag{1.3}$$

where $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} (n \in N_0)$, denotes the usual Pochhammer symbol and $_2F_1(a, b; c; z) = F(a, b; c; z)$. In our sequel we shall use the generalized Kampé de Fériet function [15,p.38] defined by

$$F_{q:q_1,\ldots,q_n}^{p:p_1,\ldots,p_n} \left[\begin{matrix} (a_p) : (b_{p_1}^1); \ldots; (b_{p_n}^n); \\ (\alpha_q); (\beta_{q_1}^1); \ldots; (\beta_{q_n}^n); \end{matrix} x_1, \ldots, x_n \right]$$
$$= \sum_{s_1,\ldots,s_n=0}^{\infty} \phi(s_1, \ldots, s_n) \frac{x_1^{s_1}}{s_1!} \cdots \frac{x_n^{s_n}}{s_n!}, \tag{1.4}$$

where

$$\phi(s_1, \cdots, s_n) = \frac{\prod_{j=1}^{p} (a_j)_{s_1+\cdots+s_n} \prod_{j=1}^{p_1} (b_j^1)_{s_1} \cdots \prod_{j=1}^{p_n} (b_j^n)_{s_n}}{\prod_{j=1}^{q} (\alpha_j)_{s_1+\cdots+s_n} \prod_{j=1}^{q_1} (\beta_j^1)_{s_1} \cdots \prod_{j=1}^{q_n} (\beta_j^n)_{s_n}} \tag{1.5}$$

for $p + p_k \leq q + q_k + 1\,(k = 1, \ldots, n)$. The symbol (a_p) means the p-dimensional vector of complex parameters $(a_1, \ldots, a_p)$ with similar interpretations for (α_q), $(a_{p_1}^1)$ etc., and it being assumed that none of the denominator parameters in (1.4) is equal to zero or negative integer.

2. Images Under Fractional Calculus Operators.

Throughout this paper we will denote $k \in A(\beta, \eta)$ if $k, \beta, \eta \in C$ satisfy the inequality

$$Re(k) > max[o, Re(\beta - \eta)] - 1.$$

We state below the formulas related to fractional calculus operators (1.1) and (1.2) which are needed in our sequel:

Lemma 1.
If $\lambda - 1 \in A(\beta, \eta)$, $p_i \leq q_i$ *and* $(p_i = q_i + 1, max|r_i x| < 1)$, $\forall i \in \{1, \ldots, n\}$, *then*

$$I_{o,x}^{\alpha,\beta,\eta}\left(x^{\lambda-1}\prod_{i=1}^{n}\left\{{}_{p_i}F_{q_i}\left[\begin{matrix}(a_{p_i}^i);\\(b_{q_i}^i);\end{matrix} r_i x\right]\right\}\right)$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda-\beta+\eta)}{\Gamma(\lambda-\beta)\Gamma(\lambda+\alpha+\eta)}x^{\lambda-\beta-1}$$
$$\cdot F_{2:q_1;\ldots;q_n}^{2:p_1;\ldots;p_n}\left[\begin{matrix}\lambda, \lambda-\beta+\eta & :(a_{p_1}^1);\ldots;(a_{p_n}^n);\\ \lambda-\beta, \lambda+\alpha+\eta & :(b_{q_1}^1);\ldots;(b_{q_n}^n);\end{matrix} r_1x, \ldots, r_nx\right] \tag{2.1}$$

Proof. Expanding each hypergeometric function and then applying the formula [12, p.55, Eqn. (2.4)]:

$$I^{\alpha,\beta,\eta}x^k = \frac{\Gamma(1+k)\Gamma(1+k-\beta+\eta)}{\Gamma(1+k-\beta)\Gamma(1+k+\alpha+\eta)}x^{k-\beta} \tag{2.2}$$

and interpreting the n-series by means of the definition (1.4), we arrive at the desired result (2.1). Similarly, we can establish the following results:

Lemma 2. *If* $\lambda - 1 \in A(\beta, \eta)$, $Re(h) > 0$, $\mu \in C$, *and* $p_i \leq q_i$ $(p_i = q_i + 1$, *Max* $|r_i x| < 1)$, $\forall i \in \{1, \ldots, n\}$, *then*

$$I_{0,x}^{\alpha,\beta,\eta}\left(x^{\lambda-1}(x+h)^{-\mu}\prod_{i=1}^{n}\left\{{}_{p_i}F_{q_i}\left[\begin{matrix}(a_{p_i}^i);\\(b_{q_i}^i);\end{matrix} r_i x\right]\right\}\right)$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda-\beta+\eta)}{\Gamma(\lambda-\beta)\Gamma(\lambda+\alpha+\beta)}x^{\lambda-\beta-1}h^{-\mu}$$
$$\cdot F_{2:q_1;\ldots;q_n;0}^{2:p_1;\ldots;p_n;1}\left[\begin{matrix}\lambda, \lambda-\beta+\eta & :(a_{p_1}^1);\ldots;(a_{p_n}^n);-\mu;\\ \lambda-\beta, \lambda+\alpha+\eta & :(b_{q_1}^1);\ldots;(b_{q_n}^n);--;\end{matrix} r_1x, \ldots, r_nx, -\frac{x}{h}\right] \tag{2.3}$$

Lemma 3. *If* $\lambda - 1 \in A(\beta, \eta)$, $Re(\mu) > 0$ *and* $p_i \leq q_i$ $(p_i = q_i + 1$, *max* $|r_i x| < 1)$, $\forall i \in \{1, \ldots, n\}$, *then*

$$I_{0,x}^{\alpha,\beta,\eta}\left(x^{\lambda-1}e^{\mu x}\prod_{i=1}^{n}\left\{{}_{p_i}F_{q_i}\left[\begin{matrix}(a_{p_i}^i);\\(b_{q_i}^i);\end{matrix} r_i x\right]\right\}\right) = \frac{\Gamma(\lambda)\Gamma(\lambda-\beta+\eta)}{\Gamma(\lambda-\beta)\Gamma(\lambda+\alpha+\eta)}x^{\lambda-\beta-1}$$
$$\cdot F_{2:q_1,\ldots,q_n}^{2:p_1,\ldots,p_n}\left[\begin{matrix}\lambda, \lambda-\beta+\eta & :(a_{p_1}^1);\ldots;(a_{p_n}^n);\\ \lambda-\beta, \lambda+\alpha+\eta & :(b_{q_1}^1);\ldots;(b_{q_n}^n);\end{matrix} r_1x, \ldots, r_nx, \mu x\right] \tag{2.4}$$

Lemma 4. *If* $\lambda - 1 \in A(\beta,\eta)$, $p_i \le q_i$, $A_j \le B_j$ *and* ($p_i = q_i + 1$, $A_j = B_j + 1$, $\max|r_i x| < 1$, $\max|s_j x| < 1$), $\forall_i \in \{1,\dots,n\}, j \in \{1,\dots,m\}$, *then*

$$I_{o,x}^{\alpha,\beta,\eta}\left(x^{\lambda-1}\prod_{i=1}^{n}\left\{{}_{p_i}F_{q_i}\left[\begin{matrix}(a_{p_i}^i);\\(b_{q_i}^i);\end{matrix} r_i x\right]\right\}\cdot\prod_{j=1}^{m}\left\{{}_{A_j}F_{B_j}\left[\begin{matrix}(c_{A_j}^j);\\(d_{B_i}^j);\end{matrix} s_j x\right]\right\}\right) \tag{2.5}$$

$$=\frac{\Gamma(\lambda)\Gamma(\lambda-\beta+\eta)}{\Gamma(\lambda-\beta)\Gamma(\lambda+\alpha+\eta)}x^{\lambda-\beta-1}F^{2:p_1;\dots;p_n;A_1;\dots;A_m}_{2:q_1;\dots;q_n;B_1;\dots;B_m}$$
$$\cdot\left[\begin{matrix}\lambda,\lambda-\beta+\eta & :(a_{p^1}^1);\dots;(a_{p_n}^n);(c_{A_1}^1);\dots;(c_{A_m}^m);\\ \lambda-\beta,\lambda+\alpha+\eta & :(b_{q^1}^1);\dots;(b_{q_n}^n);(d_{B_1}^1);\dots;(d_{B_m}^m);\end{matrix} r_1x_1,\dots,r_nx_n,s_1x_1,\dots,s_mx_m\right]$$

Remark 1. The above results (Lemmas 1-4) can also be deduced from the formulas giving the fractional calculus operator images of more involved higher classes of special functions (see[11]).

Remark 2. It may be noted that Lemma 1 can also be obtained from Lemma 2 (or Lemma 3). Lemma 3 would also follow as a limiting case of Lemma 2, and Lemma 4 can be manipulated to yield the formulas (2.1), (2.3) and (2.4).

3. A Generalized Finite Distribution

Let us define a family of distributions having the p.d.f. of the form

$$f(x)=\begin{cases}\Delta^{-1}(x-h)^{p-1}(k-x)^{\alpha-1}F\left(\alpha+\beta,-\eta;\alpha;\frac{k-x}{k-h}\right) & \\ \cdot\prod_{i=1}^{n}\left\{{}_{p_i}F_{q_i}\left[\begin{matrix}(a_{p_i}^i);\\(b_{q_i}^i);\end{matrix} r_i(k-h)\right]\right\} & \text{for } h\le x\le k,\\ 0, & \text{otherwise,}\end{cases} \tag{3.1}$$

$Re(\alpha) > 0$, $\beta,\eta \in C$, $p-1 \in A(\beta,\eta)$ and $\max_{1\le j\le n}|r_j(k-h)| < 1$ $(h \ne k)$,

where

$$\Delta=\frac{\Gamma(\alpha)\Gamma(p)\Gamma(p-\beta+\eta)}{\Gamma(p-\beta)\Gamma(p+\alpha+\eta)}(k-h)^{\alpha+p-1}$$
$$\cdot F^{2;p_1;\dots;p_n}_{2;q_1;\dots;q_n}\left[\begin{matrix}p,p-\beta+\eta & :(a_{p_1}^1);\dots;(a_{p_n}^n);\\ p-\beta,p+\alpha+\eta & :(b_{q_1}^1);\dots;(b_{q_n}^n);\end{matrix} r_1(k-h),\cdots,r_n(k-h)\right]. \tag{3.2}$$

We may mention here that the statistical distributions involving the Gaussian hypergeometric function ${}_2F_1$ in the p.d.f. were earlier studied by Dyer [1] and Mathai and Saxena [7]. The use of the generalized hypergeometric function ${}_pF_q$ in statistical distributions have also been investigated, see [14]. It seems worthwhile and of importance here to describe the motivation and usefulness behind studying the family of distributions such as the one considered in (3.1) above. Many classical distributions are known to be associated with the Beta and Gamma distributions. A large number of such distribution functions involving either one or several variates are expressible in terms of hypergeometric functions of one or more variables. To consider a specific example, it is pointed out in [15] that if U_j $(j = 1, 2)$ have a Gamma distribution with space parameter ϕ_j and scalar parameter B_j, then the ratio $X = U_1/U_2$ (U_1 and U_2 are independent) is a generalized F variate, and the distribution of X is then called the generalized F distribution with the p.d.f. given by

$$f(x) = \frac{\alpha^p}{\beta(p, m-p)} \frac{x^{p-1}}{(1+\alpha x)^m} (x > 0, \alpha > 0, m > p > 0) \tag{3.3}$$

where

$$\alpha = \frac{\beta_2}{\beta_1}, p = \phi_1, m = \phi_1 + \phi_2. \tag{3.4}$$

Dyer [1] showed that if a random variable X has the p.d.f. given by (3.3), then the distribution function of such a random variable can easily be expressed in terms of the hypergeometric function ${}_2F_1$, and the distribution function $Y = X_1 + X_2$ (where X_1 and X_2 are independent variates having p.d.f. of the form (3.3)) involves the triple hypergeometric series $F_T([15, p.43, Eqn.(10)]$. One may refer to [16,p.264] for another instance of the occurrence of multiple hypergeometric series in probability theory.

Further, large cases of statistical problems involve the distributions of linear combinations of random variables, where individual variates are assumed to have particular types of distributions. For instance, if we consider the total service time required in a routine automobile engine check-up, then the total service time required is a linear combination of random variables. In order to make inferential statements about the total service time, one needs the distribution of linear combinations of random variables. These problems are closely related to fields of queueing theory and other areas of operations research, like those encountered in inter-live-birth control (see[5]). Also, in many situations joint distributions are required to be found which involve bivariate (or multivariate) special functions (see [3] and [4]). Hence, the family of distributions considered in (3.1) may thus be looked upon as a general class of finite distributions which may be used to various statistical situations as elucidated above.

To show that (3.1) represents a p.d.f., we find that

$$\int_{-\infty}^{\infty} f(x)dx = \int_h^k f(x)dx \tag{3.3}$$

$$=\Delta^{-1} \int_h^k (x-h)^{p-1}(k-x)^{\alpha-1} F(\alpha+\beta, -\eta; \alpha; \frac{k-x}{k-h}) \prod_{i=1}^n \left\{ {}_{p_i}F_{q_i} \left[\begin{matrix} (a^i_{p_i}); \\ (b^i_{q_i}); \end{matrix} r_i(x-h) \right] \right\} dx$$

A simple change of variable puts (3.3) in the operator form as

$$\int_h^k f(x)dx = \Delta^{-1}\Gamma(\alpha)\, u^{\alpha+\beta}\, I_{o,u}^{\alpha,\beta,\eta} \left(u^{p-1} \prod_{j=1}^n \left\{ {}_{p_i}F_{q_i} \left[\begin{matrix} (a^i_{p_i}); \\ (b^i_{q_i}); \end{matrix} r_i u \right] \right\} \right) \tag{3.4}$$

Applying Lemma 1, we at once arrive at the value unity on the right side showing that (3.1) is indeed a p.d.f.

4. Expectation of Function

For any function $g(x)$, the expectation of $g(x)$ with respect to the p.d.f. $f(x)$ is defined by

$$E\{g(x)\} = \int_{-\infty}^{\infty} f(x)\, g(x)dx. \tag{4.1}$$

Consider the function $g(x)$ in terms of products of m-generalized hypergeometric functions given by

$$g(x) = \prod_{j=1}^{m} \left\{ {}_{A_j}F_{B_j} \left[\begin{matrix} (c^j_{A_j}); \\ (d^j_{B_j}); \end{matrix} s_j(x-h) \right] \right\}, \tag{4.2}$$

with $\max_{1\leq j\leq m} |s_j(k-h)| < 1$. Let a p.d.f. $f(x)$ be defined by (3.1). We have then

$$E\{g(x)\} = \Delta^{-1}\Gamma(\alpha)u^{\alpha+\beta} I_{0,u}^{(\alpha,\beta,\eta)} \left(u^{p-1} \prod_{i=1}^{n} \left\{ {}_{p_i}F_{q_i} \left[\begin{matrix} (a^i_{p_i}); \\ (b^i_{q_i}); \end{matrix} r_i u \right] \right\} \cdot \prod_{j=1}^{m} \left\{ {}_{A_j}F_{B_j} \left[\begin{matrix} (c^j_{A_j}); \\ (d^j_{B_j}); \end{matrix} s_j u \right] \right\} \right) \tag{4.3}$$

where $u = k - h$. An application of Lemma 4 yields

$$E\{g(x)\} = \nabla^{-1} F^{2:p_1;\ldots;p_n;A_1;\ldots;A_m}_{2:q_1;\ldots;q_n;B_1;\ldots;B_m} \tag{4.4}$$

$$\cdot \left[\begin{matrix} p, p-\beta+\eta & : (a^1_{p_1}); \ldots ; (a^n_{p_n}); (c^1_{A_1}); \ldots ; (c^m_{A_m}); \\ p-\beta, p+\alpha+\eta & : (b^1_{q_1}); \ldots ; (b^n_{q_n}); (d^1_{B_1}); \ldots ; (d^m_{B_m}); \end{matrix} r_1u, \ldots, r_nu, s_1u, \ldots, s_mu \right]$$

$$\nabla = F^{2:p_1;\ldots;p_n}_{2:q_1;\ldots;q_n} \left[\begin{matrix} p, p-\beta+\eta & : (a^1_{p_1}); \ldots ; (a^n_{p_n}); \\ p-\beta, p+\alpha+\eta & : (b^1_{q_1}); \ldots ; (b^n_{q_n}); \end{matrix} r_1u, \ldots, r_nu \right] \tag{4.5}$$

5. Cummulative Function

The cummulative probability function for the distrbution function $F(t)$ is given by

$$F(t) = \int_{-\infty}^{t} f(x)dx \tag{5.1}$$

For the p.d.f. $f(x)$ defined by (3.1) and for $h \leq t \leq k$. we have

$$F(t) = \int_{-\infty}^{t} f(x)dx$$

$$= \Delta^{-1} \int_h^t (x-h)^{p-1}(k-x)^{\alpha-1} F\left(\alpha+\beta, -\eta; \alpha; \frac{k-x}{k-h}\right) \prod_{i=1}^{n} \left\{ {}_{p_i}F_{q_i} \left[\begin{matrix} (a^i_{p_i}); \\ (b^i_{q_i}); \end{matrix} r_i(x-h) \right] \right\} dx.$$

This gives

$$F(t) = \Delta^{-1} \int_0^{t-h} z^{p-1}(u-z)^{\alpha-1} F(\alpha+\beta,-\eta;\alpha;\frac{u-z}{u}) \cdot \prod_{i=1}^{n} \left\{ {}_{p_i}F_{q_i} \left[\begin{matrix} (a^i_{p_i}); \\ (b^i_{q_i}); \end{matrix} r_i z \right] \right\} dz, \tag{5.2}$$

where $u = k - h$. To evaluate the integral occurring on the R.H.S. of (5.2), we proceed as follows (see also [12]):

Making use of the continuation formula of the Gauss function

$$\begin{aligned} F(a,b;c;z) =& \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a,b;a+b-c+1;1-z) \\ &+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b} F(c-a,c-b;c-a-b+1;1-z), \\ & |arg(1-z)| < \pi \end{aligned} \tag{5.3}$$

and the Eulerian transformation

$$F(a,b;c;z) = (1-z)^{c-a-b} F(c-a,c-b;c;z), \quad |arg(1-z)| < \pi, \tag{5.4}$$

in the two terms of the resulting R.H.S. of (5.2). expanding the Gauss functions and then integrating term by term. we get

$$\begin{aligned} F(t) =& \Delta^{-1} \frac{\Gamma(p-\beta)\Gamma(\eta-\beta)\Gamma(p+\alpha+\eta)}{\Gamma(-\beta)\Gamma(\alpha+\eta)\Gamma(p+\eta-\beta)\Gamma(p+1)} (\frac{t-h}{u})^p F^{1:p_1;\ldots;p_n;2}_{1:q_1;\ldots;q_n;1} \\ & \cdot \left[\begin{matrix} p & : (a^1_{p_1});\ldots;(a^n_{p_n}); & -\eta-\alpha+1,\beta+1; \\ p+1 & : (b^1_{q_1});\ldots;(b^n_{q_n}); & \beta-\eta+1; \end{matrix} r_1(t-h),\ldots,r_n(t-h),\frac{t-h}{u} \right] \\ & + \nabla^{-1} \frac{\Gamma(\beta-\eta)\Gamma(p-\beta)\Gamma(p+\alpha+\eta)}{\Gamma(-\eta)\Gamma(\alpha+\beta)\Gamma(p-\beta+\eta+1)\Gamma(p)} (\frac{t-h}{u})^{p-\beta+\eta} \\ & \cdot F^{1:p_1;\ldots;p_n;2}_{1:q_1;\ldots;q_n;1} \left[\begin{matrix} p-\beta+\eta & : (a^1_{p_1});\ldots;(a^n_{p_n}); & \eta+1,-\beta-\alpha+1; \\ p-\beta+\eta+1 & : (b^1_{q_1});\ldots;(b^n_{q_n}); & -\beta+\eta+1; \end{matrix} \right. \\ & \left. r_1(t-h),\ldots,r_n(t-h),\frac{t-h}{u} \right], \end{aligned} \tag{5.5}$$

where $u = k - h$, and ∇ is given by (4.5).

Charecteristic Function

The Charecteristic function $\phi(t)$ of a random variable x with respect to a p.d.f.$f(x)$ is defind by

$$\phi(t) = E\{e^{itx}\} = \int_{-\infty}^{\infty} e^{-itx} f(x)dx. \tag{6.1}$$

For the p.d.f.$f(x)$ defined by (3.1), we have

$$\phi(t) = \Delta^{-1} \int_h^k e^{itx}(x-h)^{p-1}(k-x)^{\alpha-1} F\left(\alpha+\beta,-\eta;\alpha;\frac{k-x}{k-h}\right) \cdot \prod_{i=1}^n \left\{ {}_{p_i}F_{q_i}\left[\begin{matrix}(a^i_{p_i});\\(b^i_{q_i});\end{matrix} r_i(x-h)\right]\right\} dx. \tag{6.2}$$

A change of variable transforms (6.2) in the following operator from:

$$\phi(t) = \Delta^{-1}\Gamma(\alpha)u^{\alpha+\beta}e^{iht} I^{\alpha,\beta,\eta}_{o,u}\left(u^{p-1}e^{itu}\prod_{i=1}^n\left\{{}_{p_i}F_{q_i}\left[\begin{matrix}(a^i_{p_i});\\(b^i_{q_i});\end{matrix} r_iu\right]\right\}\right) \tag{6.3}$$

where $u = k-h(k \neq h)$. By appealing to Lemma 3, we get the characteristic function

$$\phi(t) = e^{iht} F^{2:p_i;\dots;p_n}_{2:q_i;\dots;q_n} \tag{6.4}$$

$$\cdot\left[\begin{matrix} p, p-\beta+\eta & :(a^1_{p_1});\dots;(a^n_{p_n}); \\ p-\beta, p+\alpha+\eta & :(b^1_{q_1});\dots;(b^n_{q_n}); \end{matrix} r_1(k-h),\dots,r_n(k-h), it(k-h)\right]$$

$$\cdot F^{2:p_1;\dots;p_n}_{2:q_1;\dots;q_n}\left[\begin{matrix} p, p-\beta+\eta & :(a^1_{p_1});\dots;(a^n_{p_n}); \\ P-\beta, p+\alpha+\eta & :(b^1_{q_1});\dots;(b^n_{q_n}); \end{matrix} r_1(k-h),\dots,r_n(k-h)\right]^{-1}$$

7. Multivariate Distribution

In this concluding section we present an extended form of the Dirichlet distribution [3,p.222]. We know that the Liouville's extension of the Dirichlet's result (see[2]) is

$$\int_\Omega \cdots \int x_1^{p_1-1}\cdots x_n^{p_n-1}\phi(x_1+\ldots+x_n)dx_1\ldots dx_n = B(p_1,\dots,p_n)\int_\alpha^\beta z^{\Sigma p_i-1}\phi(z)dz, \tag{7.1}$$

for any integrable function $\phi(z)$, where the integrated region Ω is Ω: $\alpha \leq x_1+\ldots+x_n \leq \beta(x_i \geq 0; i=1,\dots,n)$, $0<\alpha<\beta$. The symbol $\sum$ denotes the summation with respect to i through 1 to n, and B means the generalized beta function:

$$B(p_1,\dots,p_n) = \frac{\Gamma(p_1)\ldots\Gamma(p_n)}{\Gamma(p_1+\ldots+p_n)}. \tag{7.2}$$

Consider the multivariate density function

$$f(x_1,\dots,x_n) = \begin{cases} \omega x_1^{p_1-1}\ldots x_n^{p_n-1}(\sum x_i-\alpha)^{p-1}(\beta-\sum x_i)^{\alpha-1} \\ .F\left(\lambda+\mu,-\varrho;\lambda;\frac{\beta-\sum x_i}{\beta-\alpha}\right)\prod_{i=1}^n\left\{{}_{p_i}F_{q_i}\left[\begin{matrix}(a^i_{p_i});\\(b^i_{q_i});\end{matrix}\sum x_i-\alpha\right]\right\}, \\ \qquad \text{for} \quad \alpha\leq\sum x_i\leq\beta \\ 0, \quad \text{elsewhere,}\end{cases} \tag{7.3}$$

where $Re(p_i) > 0$ $(i = 1, \ldots, n)$, $Re(p) > 0, \lambda, \mu, p \in C$, $p - 1 \in A(\mu, p)$, and

$$\omega^{-1} = B(p_1, \ldots, p_n) \frac{\Gamma(\lambda)\Gamma(p)\Gamma(p-\mu+p)}{\Gamma(p-\mu)\Gamma(p+\lambda+p)} u^{p+\lambda-1} F^{2:p_1;\ldots;p_n}_{2:q_1;\ldots;q_n} \left[\begin{matrix} p, p-\mu+p & :(a^1_{p_1});\ldots;(a^n_{p_n}); & 1-\sum p_i; \\ p-\mu, p+\alpha+p & :(b^1_{q_1});\ldots;(b^n_{q_n}); & -----; \end{matrix} u, \ldots, u, -\frac{u}{\alpha} \right] \tag{7.4}$$

where $u = \beta - \alpha$. To verify that (7.3) represents a p.d.f.,we observe that

$$\begin{aligned} &\int_\Omega \int f(x_1, \ldots, x_n) dx_1 \ldots dx_n \\ =&\omega \int \cdots \int_{x_1 \geq, \alpha \leq \sum x_i \leq \beta} x_1^{p_1-1} \ldots x_n^{p_n-1} (\sum x_i - \alpha)^{p-1} (\beta - \sum x_i)^{\lambda-1} \\ &\cdot F(\lambda+\mu, -\rho; \lambda; \frac{\beta - \sum x_i}{\beta - \alpha}) \prod_{i=1}^n \left\{ {}_{p_i}F_{q_i} \left[\begin{matrix} (a^i_{p_i}); \\ (b^i_{q_i}); \end{matrix} \sum x_i - \alpha \right] \right\} dx_1 \ldots dx_n \\ =&\omega B(p_1 \ldots, p_n) \int_\alpha^\beta z^{\Sigma p_i - 1} (z-\alpha)^{p-1} (\beta - z)^{\lambda-1} F\left(\lambda+\mu, -\rho : \lambda; \frac{\beta - z}{\beta - \alpha}\right) \\ &\cdot \prod_{i=1}^n \left\{ {}_{p_i}F_{q_i} \left[\begin{matrix} (a^i_{p_i}); \\ (b^i_{q_i}); \end{matrix} z - \alpha \right] \right\} dz \\ =&\omega B(p_1, \ldots, p_n) \Gamma(\lambda) u^{\lambda+\mu} I^{\lambda,\mu,\rho}_{o,u} \left(u^{p-1} (u+\alpha)^{\sum p_i - 1} \cdot \prod_{i=1}^n \left\{ {}_{p_i}F_{q_i} \left[\begin{matrix} (a^i_{p_i}); \\ (b^i_{q_i}); \end{matrix} u \right] \right\} \right) \end{aligned} \tag{7.5}$$

where $u = \beta - \alpha$. In view of Lemma 2, we find that the value of R.H.S. of (7.5) is unity by virtue of (7.4). This shows that (7.3) is a p.d.f.

Several other properties of statistical nature can be considered for the multivariate distribution characterised by (7.3), as indicated in [4]. In the concluding remark, it may be mentioned that various useful results in statistical distributions and related problems investigated in [3], [4] and [12] can be deduced from the results presented in this paper.

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