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# A NOTE ON A FIXED POINT PROPERTY FOR METRIC PROJECTIONS

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**Abstract.** The paper contains a partial answer to a question rasied recently by S.P. Singh concerning the existence of fixed points of metric projections.

Let $E$ be a real Banach space with the norm $\|\cdot\|$. Recall that a subset $X$ of $E$ is said to be a Chebyshev if for every $x \in E$ there exists a unique $z \in X$ such that $\|x-z\|$ =dist$(x, X)$. In this case we can define the so-called metric projection $\mathcal{P}_X$ of $E$ onto $X$ which assigns to each $x \in E$ the point $z \in X$ such that $\|x-z\|$ =dist$(x, X)$.

It is well-known [3] that if $E$ is reflexive and strictly convex then every clsoed convex subset $X$ of $E$ is a Chebyshev set. Thus for every closed convex subset $X$ of a reflexive and strictly convex Banach space $E$ we can define the metric projection $\mathcal{P}_X : E \to X$.

During the conference "Functional Analysis and Applications" held at Gargnano del Garda, Italy (10-14 May 1993) professor S.P. Singh raised the following problem:

Let $A$, $B$ be Chebyshev sets in a real Banach space $E$ and let $P_A$, $P_B$ be the metric projections of $E$ onto the sets $A$ and $B$, respectively. Consider the mapping $P_AP_B$: $A \to A$. Does there exist a fixed point of this mapping?

Observe that in the case when $E$ is a real Hilbert space the answer is affirmative provided $A$ and $B$ are closed, convex and bounded subsets of $E$. It is an easy consequence of the fact that the metric projection is nonexpansive in this setting, so the well-known Browder-Gödhe-Kirk fixed point theorem gives the desired answer (cf. [2]). But it is no longer true for other Banach spaces although they have nice geometrical structure. For example, if $E$ is uniformly convex then the metric projection is only continuous [2].

Nevertheless we show that a large class of Banach spaces has fixed point property with respect to the mapping $P_AP_B$.

We define

$$\text{dist}(A, B) = \inf\{\|a-b\| : a \in A, b \in B\}.$$

Let us start with the following Lemma.

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**Lemma 1.** *Let $A$ and $B$ be Chebyshev sets in an arbitrary Banach space $E$. If there exist points $a \in A$ and $b \in B$ such that $\|a - b\| =$dist$(A, B)$ then $a$ is the fixed point of the mapping $P_A P_B$.*

The proof is trivial and is therefore omitted.

Our main result answering the question of S.P. Singh for a large class of Banach spaces is contained in the following theorem.

**Theorem 1.** *Let $E$ be a reflexive and strictly convex Banach space and let $A$, $B$ be nonempty, bounded, closed and convex subsets of $E$. Then there exist points $a \in A$ and $b \in B$ such that $\|a - b\| =$dist $(A, B)$ and simultaneously $a$ is the fixed point of the mapping $P_A P_B$.*

**Proof.** Let $\{a_n\} \subset A$ and $\{b_n\} \subset B$ be sequnces such that

$$\lim_{n \to \infty} \|a_n - b_n\| = \text{dist}(A, B).$$

In view of reflexivity of the space $E$ and the assumption on boundedness of $A$ and $B$ we may assume (taking subsequences if necessary) that $\{a_n\}$ and $\{b_n\}$ converge weakly to points $a$ and $b$, respectively. By Mazur's theorem we deduce that $a \in A$ and $b \in B$. Next, let us observe that the sequence $\{a_n - b_n\}$ converges weakly to the point $a - b$. Hence, in view of lower semicontinuity of the norm [1] we infer that

$$\|a - b\| \leq \liminf_{n \to \infty} \|a_n - b_n\| = \lim_{n \to \infty} \|a_n - b_n\|.$$

This implies that $\|a - b\| =$dist $(A, B)$.

Finally, applying Lemma 1 we complete the proof.

In order to show that the assumption on boundedness of the sets $A$ and $B$ in the above theorem is essential, consider the following example.

**Example.** Take the Euclidean plane $R^2$. Let $A$, $B$ be subsets of $R^2$ defined as follows:

$$A = \{(x, 0) \, : \, x \geq 0\},$$
$$B = \{(x, y) \, : y \geq 1/x, \, x > 0\}.$$

Obviously $A$ and $B$ are closed, convex but unbounded subsets of $R^2$. It is easily seen that the mapping $P_A P_B$ has no fixed points in the set $A$.

On the other hand dist $(A, B) = 0$ but do not exist points $a \in A$ and $b \in B$ such that $\|a - b\| = 0$.

## References

[1] M. M. Day, *Normed Linear Spaces*, Springer, Berlin-Heidelberg, New York, 1973.
[2] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geomety, and Nonexpansive Mappings*, Marcel Dekker, New York, 1984.
[3] G. Köthe, *Toplogical Vector Spaces I*, Springer, Berlin, 1969.

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