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# A NOTE ON SOME SERIES INEQUALITIES

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**Abstract.** In the present note we establish some new inequalities involving series of nonnegative terms by using a fairly elementary analysis.

A large number of papers have been written on inequalities involving series of nonnegative terms and their applications in various parts of analysis, see [1-5] and the references given therein. The main purpose of this note is to prove the following.

**Theorem.** *Let $p \geq 1$, $q \geq 1$, $r \geq 1$ be real constants. If $a_n \geq 0$, $(n = 1, 2, \ldots)$ and $A_n = \sum_{m=1}^{n} a_m$, then*

$$\sum_{n=1}^{N} a_n A_n \leq \frac{(N+1)}{2} \sum_{n=1}^{N} a_n^2, \tag{1}$$

$$\sum_{n=1}^{N} A_n^{p+q} \leq [(p+q)(N+1)]^q \sum_{n=1}^{N} A_n^p a_n^q, \tag{2}$$

$$\sum_{n=1}^{N} A_n^{p+q} a_n^r \leq [(p+q+r)(N+1)]^q \sum_{n=1}^{N} A_n^p a_n^{q+r}. \tag{3}$$

**Proof.** Rewriting the left side of (1), and using the Schwarz inequality, interchanging the order of summations, and using the elementary inequality $a^{1/2} b^{1/2} \leq \frac{1}{2}(a+b)$, $(a \geq 0,\ b \geq 0$ reals), we observe that

$$\begin{aligned} \sum_{n=1}^{N} a_n A_n &= \sum_{n=1}^{N} (\sqrt{n} a_n)\Big(\frac{1}{\sqrt{n}} \sum_{m=1}^{n} a_m\Big) \\ &\leq \Big[\sum_{n=1}^{N} n\, a_n^2\Big]^{1/2} \Big[\sum_{n=1}^{N} \frac{1}{n}\Big(\sum_{m=1}^{n} a_m\Big)^2\Big]^{1/2} \end{aligned}$$

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$$\leq [\sum_{n=1}^{N} n\, a_n^2]^{1/2}[\sum_{n=1}^{N} \frac{1}{n}(\sum_{m=1}^{n} 1)(\sum_{m=1}^{n} a_m^2)]^{1/2}$$
$$= [\sum_{n=1}^{N} n\, a_n^2]^{1/2}[\sum_{n=1}^{N}(\sum_{m=1}^{n} a_m^2)]^{1/2}$$
$$= [\sum_{n=1}^{N} n\, a_n^2]^{1/2}[\sum_{m=1}^{N} a_m^2(\sum_{n=m}^{N} 1)]^{1/2}$$
$$= [\sum_{n=1}^{N} n\, a_n^2]^{1/2}[\sum_{m=1}^{N}(N-m+1)a_m^2]^{1/2}$$
$$\leq \frac{1}{2}[\sum_{n=1}^{N} n\, a_n^2 + \sum_{n=1}^{N}(N-n+1)a_n^2]$$
$$= \frac{(N+1)}{2}\sum_{n=1}^{N} a_n^2.$$

This proves the required inequality in (1).

By taking $z_m = a_m$ and $\alpha = p+q$ in the following inequality (see,[2,5])

$$(\sum_{m=1}^{n} z_m)^\alpha \leq \alpha \sum_{m=1}^{n} z_m(\sum_{k=1}^{m} z_k)^{\alpha-1},$$

where $\alpha > 1$ is a constant and $z_m \geq 0$, $(m = 1, 2, \ldots)$, we have

$$A_n^{p+q} \leq (p+q)\sum_{m=1}^{n} a_m A_m^{p+q-1}. \tag{4}$$

By taking the sum on both sides of (4) from 1 to $N$ and interchanging the order of the summation, we observe that

$$\sum_{n=1}^{N} A_n^{p+q} \leq (p+q)\sum_{n=1}^{N}(\sum_{m=1}^{n} a_m A_m^{p+q-1})$$
$$= (p+q)\sum_{m=1}^{N} a_m\, A_m^{p+q-1}(N-m+1)$$
$$\leq (p+q)(N+1)\sum_{m=1}^{N} a_m A_m^{p+q-1}$$
$$= (p+q)(N+1)\sum_{n=1}^{N}(A_n^{p/q} a_n)(A_n^{p+q-1-p/q}). \tag{5}$$

By using the Hölder's inequality with indices $q$, $q/(q-1)$ on the right side of (5) we have

$$\sum_{n=1}^{N} A_n^{p+q} \leq (p+q)(N+1)[\sum_{n=1}^{N} A_n^p a_n^q]^{1/q}[\sum_{n=1}^{N} A_n^{p+q}]^{(q-1)/q}. \tag{6}$$

Dividing by the last factor on the right side of (6) and raising to the $q$th power of the resulting inequality, we get the desired inequality in (2).

By rewriting the left side of (3) and using the Hölder's inequality with indices $(q+r)/r$, $(q+r)/q$ and the inequality (3), we observe that

$$\begin{aligned}\sum_{n=1}^{N} A_n^{p+q} a_n^r &= \sum_{n=1}^{N}(A_n^{(pr)/(q+r)} a_n^r)(A_n^{p+q-(pr)/(q+r)}) \\ &\leq [\sum_{n=1}^{N} A_n^p a_n^{q+r}]^{r/(q+r)}[\sum_{n=1}^{N} A_n^{p+q+r}]^{q/(q+r)} \\ &\leq [\sum_{n=1}^{N} A_n^p a_n^{q+r}]^{r/(q+r)}[[(p+q+r)(N+1)]^{q+r}\sum_{n=1}^{N} A_n^p a_n^{q+r}]^{q/(q+r)} \\ &= [(p+q+r)(N+1)]^q \sum_{n=1}^{N} A_n^p a_n^{q+r}.\end{aligned}$$

This is the required inequality in (3), and the proof of the theorem is complete.

It is interesting to note that by taking $p = 1$, $q = 1$ in (2) we can obtain the lower bound on the left side of the inequality given in (1). For various other inequalities involving series of terms, see [1-5] and the references given therein.

## References

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