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A NOTE ON HUA'S INEQUALITY FOR COMPLEX NUMBERS

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Abstract. In this paper we give a general Hua's inequality involving a finite sequence of complex numbers.

1. Introduction

The following inequality of Lo-Keng Hua [1, P.149] is important in Number Theory.

Theorem 1. *Let* δ, $\alpha > 0$ *and* $x_i \in R (i = 1, \cdots, n)$. *Then*

$$(\delta - \sum_{i=1}^{n} x_i)^2 + \alpha \sum_{i=1}^{n} x_i^2 \geq \frac{\alpha}{n+\alpha}\delta^2 \tag{1}$$

with equality if and only if

$$x_i = \frac{\delta}{n+\alpha}, \ i = 1, \cdots, n. \tag{2}$$

The following generalization of Theorem 1 is due to Chung-Lie Wang [2]. See also [5].

Theorem 2. *Let* δ, $\alpha > 0$, *and*

$$F_n(x) = (\delta - \sum_{i=1}^{n} x_i)^p + \alpha^{p-1}(\sum_{i=1}^{n} x^p). \tag{3}$$

Then for $p > 1$, *the inequality*

$$F_n \geq (\frac{\alpha}{n+\alpha})^{p-1}\delta^p \tag{4}$$

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holds for all nonnegative $x_i \in \mathbf{R}(i = 1, 2, 3, \cdots, n)$ *with* $\sum_{i=1}^{n} x_i \leq \delta$.

The sign of enequality in (4) is reversed for $0 < p < 1$. *In either case, the sign of equality holds if and only if (2) holds.*

Recently, Dragomir [3] established an analogue inequality of (1) for complex numbers as follow.

Theorem 3. *Let* $\alpha > 0$ *and* $\delta, z_1, z_2, \cdots, z_n \in \mathbf{C}$. *Then the following inequality*

$$|\delta - \sum_{i=1}^{n} z_i|^2 + \alpha \sum_{i=1}^{n} |z_i|^2 \geq \frac{\alpha}{n+\alpha} |\delta|^2 \tag{5}$$

holds. The equality is valid in (5) if and only if

$$z_i = \frac{\delta}{n+\alpha}, \quad i = 1, 2, \cdots, n. \tag{6}$$

The main purpose of this note is to give a generalization of Theorem 3.

Gneralization of theorem 3

Theorem 4. *If* $\alpha > 0$, *let* $\delta, z_1, \cdots, z_n \in \mathbf{C}$ *and*

$$F_n(z) = \left|\delta - \sum_{i=1}^{n} z_i\right|^p + \alpha^{p-1} \sum_{i=1}^{n} |z_i|^p$$

The for $p > 1$, *we have*

$$F_n(z) \geq (\frac{\alpha}{n+\alpha})^{p-1} |\delta|^p, \tag{7}$$

and the sign of equality holds in (16) if and only if

$$z_i = \frac{\delta}{n+\alpha}, \qquad i = 1, \cdots, n.$$

Proof. we need the following lemma.

Lemma 4.1. [4, P.26]: *If* $k > 1$, *and* k' *is the conjugate of* k, *then*

$$\left|\sum ab\right| \leq (\sum |a|^{k'})^{\frac{1}{k'}} (\sum |b|^k)^{\frac{1}{k}} \tag{8}$$

holds for any finite complex sequences (a), (b). *There is equality if and only if* $(|a_v|^{k'})$ *and* $(|b_v|^k)$ *are proportional and* $\arg a_v b_v$ *is independent of* v.

Now, let $\lambda = \delta - \sum_{i=1}^{n} z_i$. Then $z_1 = \delta - \lambda - \sum_{i=2}^{n} z_i$. Applying (8) with $k = p > 1$ to the following two sequences $(a), (b)$:

$$a_1 = 1, b_1 = \lambda, a_2 = \alpha^{\frac{1-p}{p}}, b_2 = (\delta - \lambda - \sum_{i=2}^{n} z_i)\alpha^{\frac{p-1}{p}},$$

and

$$a_i = \alpha^{\frac{1-p}{p}}, b_i = z_{i-1}\alpha^{\frac{p-1}{p}} \text{ for } 3 \le i \le n+1.$$

where $\sum_{i=1}^{n+1} a_i b_i = \delta$, we have

$$\begin{aligned}|\delta \le &(1+n\alpha^{-1})^{\frac{p-1}{p}}(|\lambda|^p + \left|\delta - \lambda - \sum_{i=2}^{n}\right|^p \alpha^{p-1} + \sum_{i=2}^{n}|z_i|^p\alpha^{p-1})^{\frac{1}{p}}\\ =&(\frac{n+\alpha}{\alpha})^{\frac{p-1}{p}}(F_n(z))^{\frac{1}{p}},\end{aligned}$$

so that

$$|\delta|^p \le (\frac{n+\alpha}{\alpha})^{p-1}F_n(z),$$

which is equivalent to

$$F_n(z) \ge (\frac{\alpha}{n+\alpha})^{p-1}|\delta|^P. \tag{9}$$

If the equality holds in (9), then

$$\frac{|\lambda|}{1} = \frac{|\delta - \lambda - \sum_{i=2}^{n} z_i|}{\alpha^{-1}} = \frac{|z_2|}{\alpha^{-1}} = \cdots = \frac{|z_n|}{\alpha^{-1}} \text{ with } z_1 = \delta - \lambda - \sum_{i=2}^{n} z_i, \tag{10}$$

and

$$\delta - \lambda - \sum_{i=2}^{n} z_i = t_1\lambda \text{ ; } z_i = t_i\lambda,\ t_i \in \mathbf{R}_+, i = 2\cdots, n. \tag{11}$$

It follows from (10) and (11) that

$$z_i = \frac{\delta}{n+\alpha}, \quad i = 1, \cdots, n.$$

Converserly, if

$$z_i = \frac{\delta}{n+\alpha},\ i = 1, 2, \cdots, n,$$

it is easily seen that

$$F_n(z) = (\frac{\alpha}{n+\alpha})^{p-1}|\delta|^p.$$

This completes the proof.

Corollary. *If $\alpha > 0$, $\delta \in C$ and (w_i) is a sequence of complex numbers such that $\sum w_i$ is convergent, then for $p > 1$, we have*

$$\left|\delta - \sum w_i\right|^p + \alpha^{p-1}|\sum w_i|^p \ge (\frac{\alpha}{1+\alpha})^{p-1}|\delta|^p. \tag{12}$$

The equality holds in (12) *if and only if*

$$\sum w_i = \frac{\delta}{1+\alpha}.$$

Proof. Let $z_1 = \sum w_i$. Then applying Theorem 4 to the case $n = 1$.

Remark. It is not necessarily true that the sign of the inequality in (7) is reversed for $0 < p < 1$.

For example, let

$$p = \frac{1}{2}, \alpha = \delta = 1, z_1 = i \text{ and } z_3 = -1.$$

Then we have

$$F_3(z) = \left|1 - \sum_{k=1}^{3} z_k\right|^{\frac{1}{2}} = 3$$
$$> 2 = (\frac{1}{3+1})^{-\frac{1}{2}}.$$

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