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THE FAMILY OF FUNCTIONS $S_{\alpha,k}$ AND THE LIÉNARD EQUATION

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Abstract. In this paper we study qualitatively the Liénard Equation $\ddot{x}+f(x)\dot{x}+g(x)=0$ with aid of the non-usual family of functions given by

$$S_{\alpha,k}(x,y)=\int_0^{y+F(x)-\alpha G(x)-k}\frac{s}{\alpha s+1}dx+\int_0^x g(u)du$$

where $F(x)=\int_0^x f(u)du.G(x)=\int_0^x g(u)du$ and $\alpha, k \int R$.

1. Introduction and Prelimanaries

Throughout this work we consider the equation

$$\ddot{x}+f(x)\dot{x}+g(x)=0 \tag{1}$$

where f and g are functions of $\mathbb{R}$ in $\mathbb{R}$ satisfying the following conditions:
a) f and g are continuous and ensure uniqueness of solutions.
b) $x.g(x)>0$ for $x\neq 0$.
Next, we suppose the above conditions are verified and they will not be mentioned again.

The equation (1) is equivalent to the system

$$\begin{cases}\dot{x}=y\\ \dot{y}=-f(x)y-g(x)\end{cases} \tag{2}$$

The condition b) ensures the orign (0,0) is the only singular point of (2).

The more natural positive definite function for studyng qualitatively the system (2) is the Energy Function

$$E(x,y)=\frac{1}{2}y^2+\int_0^x g(u)du$$

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whose derivative relative to (2) is $\dot{E}(x,y) = -f(x)y^2$.

In [1] we study qualitatively (2) using the family of positive definite functions given by

$$V_\alpha(x,y) = \int_0^y \frac{s}{\alpha s+1} ds + \int_0^x g(u)du$$

where V_0 is exactly the Energy Function.

The equation (1) is also equivalent to the system

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x) \end{cases} \tag{3}$$

where $F(x) = \int_0^x f(u)du$. In several works (for example [2], [3] and [4] the system (3) was studied with aid of the family of functions

$$E_k(x,y) = \frac{1}{2}(y-k)^2 + \int_0^x g(u)du$$

whose derivative relative to the system (3) is $\dot{E}_k(x,y) = -g(x)[F(x) - k]$.

Condsider now the function

$$S_{0,k} = \frac{1}{2}[y + F(x) - k]^2 + \int_0^x g(u)du.$$

The derivative of $S_{0,k}$ relative to the system (2) is $\dot{S}_{0,k}(x,y) = -g(x)[F(x) - k]$.

So the function $S_{0,k}$ plays, relatively to the system (2), the same role that E_k relatively to (3). The function $S_{0,k}$ is a member ($\alpha = 0$) of family $S_{\alpha,k} : \Omega_{\alpha,k} \to \mathbb{R}$ given by

$$S_{\alpha,k}(x,y) = \int_0^{y+F(x)-\alpha G(x)-k} \frac{s}{\alpha S+1} ds + \int_0^x g(u)du$$

where $F(x) = \int_0^x f(u)du$, $G(x) = \int_0^x g(u)du$ and $\Omega_{\alpha,k}$ is the following open set:

$$\Omega_{\alpha,k} = \mathbb{R}^2 \quad \text{if } \alpha = 0$$

$$\Omega_{\alpha,k} = \left\{(x,y) \in \mathbb{R}^2 \mid y > -F(x) + \alpha G(x) + k - \frac{1}{\alpha}\right\} \text{ if } \alpha > 0$$

and

$$\Omega_{\alpha,k} = \left\{(x,y) \in \mathbb{R}^2 \mid y < -F(x) + \alpha G(x) + k - \frac{1}{\alpha}\right\} \text{ if } \alpha < 0$$

The derivative of $S_{\alpha,k}$ relative to the system (2) is

$$\dot{S}_{\alpha,k}(x,y) = -\frac{g(x)[F(x) - \alpha G(x) - k]}{\alpha[y + F(x) - \alpha G(x) - k] + 1}. \tag{4}$$

We observe that the sign of $\dot{S}_{\alpha,k}$ is the same of $-g(x)[F(x)-\alpha G(x)-k]$ because $\alpha[y+F(x)-\alpha G(x)-k]+1>0$ on $\Omega_{\alpha,k}$.

In this work we study qualitatively (2) utilizing the family of functions $S_{\alpha,k}$. We observe that the new idea in this paper is only the family $S_{\alpha,k}$. However, as we shall see, the level curves of this family and the relation (4) together suggest to us how to state, in a natural way, several qualitative results about the solutions of the Liénard Equation.

The system (2) can be also studied using the family of positive definite functions given by

$$W_{\alpha,\beta}(x,y)=\int_0^{y/H_\beta(x)}\frac{s}{s^2+\alpha s+1}ds+\ln\beta^{-1/2}H_\beta(x)$$

where $H_\beta(x)=[2G(x)+\beta]^{1/2}$, $\beta>0$ (see[5]).

It is clear that we can also study qualitatively the system (2) combining the functions V_α, $W_{\alpha,\beta}$ and $S_{\alpha,k}$. Many interesting and important works about the Leénard Equation have been published and some are listed in the references. I have a special caress by Theorem 2 in [7], because with aid of it (and of a dream!) I concluded my Doctoral Thesis and indirectly my work [5] was suggested by it.

2. Auxiliary Lemmas

Next, we suppose $\alpha\geq 0$ and

$$w(x)=-F(x)+\alpha G(x)+k-\frac{1}{\alpha}\quad\text{if } \alpha>0 \quad\text{and}\quad w(x)=-\infty \quad \text{if } \alpha=0.$$

In first place we observe that, for each fixed x, the the function

$$y\mapsto S_{\alpha,k}(x,y)$$

is strictly increasing for $y\geq -F(x)+\alpha G(x)+k$ and strictly decreasing for $w(x)<y<-F(x)+\alpha G(x)+k$. We have also

$$\lim_{y\to+\infty}S_{\alpha,k}(x,y)=+\infty=\lim_{y\to w(x)+}S_{\alpha,k}(x,y).$$

So for each $c>0$ and for each x, with $G(x)<c$, there is a unique $y_1>-F(x)+\alpha G(x)+k$ and a unique y_2, with $u(x)<y_2<-F(x)+\alpha G(x)+k$ such that

$$S_{\alpha,k}(x,y_1)=S_{\alpha,k}(x,y_2)=c$$

. If there exist $x_1<0<x_2$ such that

$$G(x_1)=G(x_2)=c$$

then the level curve $S_{\alpha,k}(x,y)=c$ is clsoed and shows, in the case $F(x)\geq\alpha G(x)+k$ and $k<0$ and $c>\int_0^{-k}\frac{s}{\alpha s+1}ds$, the following aspect (Figure 1):

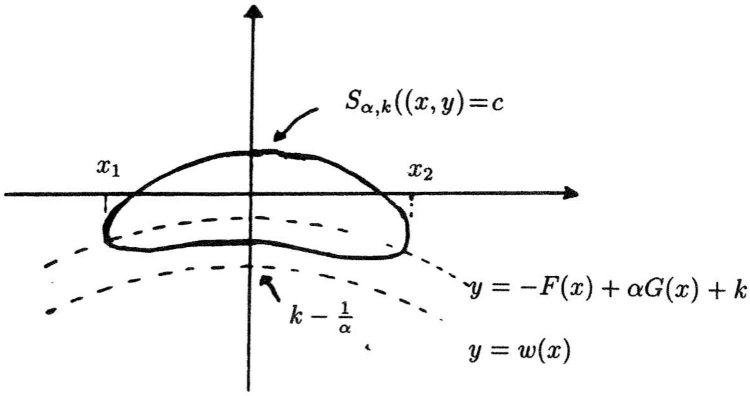

Figure 1

If $F(x) \geq \alpha G(x) + k$ for $x \geq 0$ and there exists $x_2 > 0$ such that $S_{\alpha,k}(x_2, 0) = c$ then the arc

$$S_{\alpha,k}(x,y) = c,$$

with $x > 0$ and $y > -F(x) + \alpha G(x) + k$, crosses the $x > 0$ half-axis at $(x_2, 0)$ and the set

$$\{(x,y) \in \Omega_{\alpha,k} \mid S_{\alpha,k}(x,y) = c, \quad 0 \leq x \leq x_2\}$$

shows, in the case $k < 0$ and $S_{\alpha,k}(x,0) < c$, for $0 < x < x_2$, the following aspect (Figure 2):

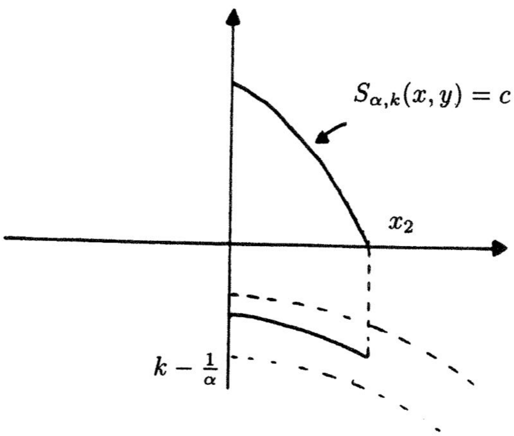

Figure 2

If $F(x) \leq \beta G(x) + k$, for $\alpha \leq x \leq 0$, with $k - \frac{1}{\beta} < 0$, and $\omega(\alpha) > 0$, then the curve

$$y = \omega(x) = -F(x) + \beta G(x) + k - \frac{1}{\beta}$$

crosses the $x < 0$ half-axis. Hence, for every $c > 0$, the arc

$$S_{\alpha,k}(x,y) = c, \quad x < 0 \text{ and } y < -F(x) + \beta G(x) + k$$

crosses too the $x < 0$ half-axis at point $(x_1, 0)$, with $a < x_1 < 0$ and the set

$$\{(x,y) \in \Omega_{\beta,k} | S_{\beta,k}(x,y) = c, \ x_1 \leq x \leq 0\}$$

shows, in the case $k > 0$ and $w(x) < 0$ for $x_2 < x < 0$, the following aspect (Figure 3):

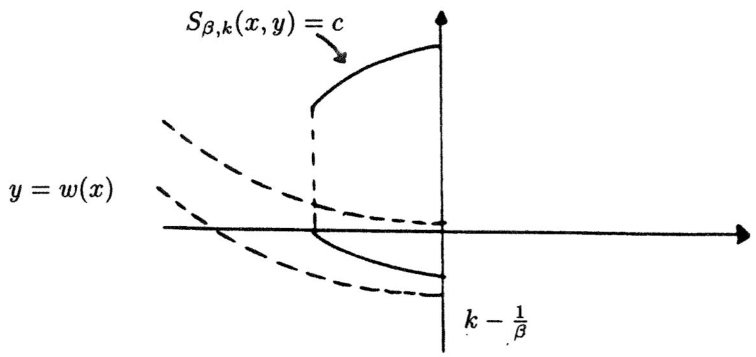

Figure 3

If $\alpha = 0$ we have

$$S_{0,k}(x,y) = \frac{1}{2}(y + F(x) - k)^2 + G(x).$$

So, $S_{0,k}(x,y) = c$ is equivalent to

$$y = -F(x) + k + [2c - 2G(x)]^{1/2} \text{ or } y = -F(x) + k - [2c - 2G(x)]^{1/2}$$

The case $\alpha < 0$ can be discussed in a similar way.

Lemma 2.1 *Suppose there are $\alpha > 0$, $b > 0$ and $k \leq 0$ such that*

$$F(x) \geq \alpha G(x) + k \quad \textit{for } 0 \leq x \leq b. \tag{5}$$

Let $\gamma(t) = (x(t), y(t))$ *be the solution of* (2) *with* $\gamma(0) = (x_0, 0)$, $0 < x_0 \le b$, *and* $t_1 > 0$ *such that* $0 \le x(t) \le b$ *for* $0 \le t \le t_1$, *Then, for* $0 \le t \le t_1$,

$$y(t) > -F(x(t)) + \alpha G(x(t)) + k - \frac{1}{\alpha}.$$

In particular, if $x(t_1) = 0$ *then* $y(t_1) > k - \frac{1}{\alpha}$.

Proof. From (5), $\dot{S}_{\alpha,k}(x,y) \le 0$, $0 \le x \le b$. It follows that, for each $u \in]0, t_1]$ such that $\gamma(t) \in \Omega_{\alpha,k}$, $0 \le t \le u$, we have $\dot{S}_{\alpha,k}(\gamma(t)) \le 0$ for $0 \le t \le u$, and therefore

$$S_{\alpha,k}(\gamma(t)) < c, \qquad 0 \le t \le u,$$

with $c > S_{\alpha,k}(\gamma(0))$. It follows immediately that the set $\{\gamma(t) \mid 0 \le t \le t_1\}$ does not intercept the arc

$$S_{\alpha,k}(x,y) = C, \qquad y \le F(x) + \alpha G(x) + k.$$

Then, for $0 \le t \le t_1$ we have

$$y(t) > -F(x(t)) + \alpha G(x(t)) + k - \frac{1}{\alpha}.$$

(This result is intuitive: it is enough to look the Figure 2 with $x_0 < x_2$)

Lemma 2.2 *Suppose there are* $b > 0$, $\alpha \ge 0$, $R > 0$ *and* $k \le 0$ *such that*

$$F(x) \ge \alpha G(x) + k, \quad 0 \le x \le b \tag{6}$$

$$F(b) \ge k + [(R - k)^2 - 2G(b)]^{1/2}. \tag{7}$$

Let $\gamma(t)$ *be the solution of (2) with* $\gamma(0) = (0, y_0)$, $0 < y_0 \le R$. *Then there is* $t_1 > 0$ *such that*

$$\gamma(t_1) = (b_1, 0), \quad 0 < b_1 \le b.$$

Moreover, if there exists $t_2 > t_1$ *such that* $0 \le x(t) \le b$ *for* $t_1 \le t \le t_2$ *and* $x(t_2) = 0$ *then*

$$y(t_2) > k - \frac{1}{\alpha} \text{ if } \alpha > 0 \text{ and } y(t_2) \ge -R + 2k \text{ if} \alpha = 0.$$

Proof. The equation

$$S_{0,k}(x,y) = S_{0,k}(0,R) = \frac{1}{2}(R-k)^2$$

is equivalent to

$$y = -F(x) + k + [(R-k)^2 - 2G(x)]^{1/2} \tag{8}$$

or

$$y = -F(x) + k - [(R-k)^2 - 2G(x)]^{1/2} \tag{9}$$

The condition (7) ensures the curve (8) intercepts the $x > 0$ half-axis at a point $(b_2, 0)$, $0 < b_2 \le b$. Form (6) have $F(x) \ge k$, $0 \le x \le b$. So

$$\dot{S}_{0,k}(x,y) \le 0, \quad 0 \le x \le b. \tag{10}$$

Then the solution $\gamma(t)$ of (2) starting at the point $\gamma(0) = (0, y_0)$, $0 < y_0 \le R$, crosses also the $x > 0$ half-axis at a point $\gamma(t_1) = (b_1, 0)$, $0 < b_1 \le b_2 \le b$. From (10) we have for $t_1 \le t \le t_2$

$$S_{0,k}(\gamma(t)) \le S_{0,k}(\gamma(t_1)) \le S_{0,k}(0,R).$$

Hence and from (9) we have for $t_1 \le t \le t_2$

$$y(t) \ge -F(x(t)) + k - [(R-k)^2 - 2G(x(t))]^{1/2}.$$

So, if $\alpha = 0$ and $x(t_2) = 0$ we have $y(t_2) \le -R + 2k$.
From Lemma 2.1, if $\alpha > 0$,

$$y(t_2) > k - \frac{1}{\alpha}.$$

(See again Figure 2.)

Lemma 2.3 *Suppose there are $a < 0$, $\beta > 0$, $R < 0$ and $k \ge 0$ such that*

$$F(x) \le \beta G(x) + k, \quad a \le x \le 0, \tag{11}$$

$$F(a) \le \beta G(a) + k - \frac{1}{\beta} \tag{12}$$

and

$$R > k - \frac{1}{\beta}. \tag{13}$$

Let $\gamma(t) = (x(t), y(t))$ be the solution of (2) with $\gamma(0) = (0, y_0)$, $R \le y_0 < 0$. Then there is $t_1 > 0$ such that $\gamma(t_1) = (a_1, 0)$ with $a \le a_1 < 0$, and $y(t) > -F(x(t)) + \beta G(x(t)) + k - \frac{1}{\beta}$, $0 \le t \le t_1$.

Proof. The conditions (12) and (13) ensure the curve

$$y = -F(x) + \beta G(x) + k - \frac{1}{\beta}$$

crosses the $x < 0$ half-axis at a point $(a_2, 0)$ with $a \le a_2 < 0$. From (11)

$$\dot{S}_{\beta,k} \le 0, \qquad a \le x \le 0.$$

So, the solution $\gamma(t)$ starting $\gamma(0) = (0, y_0)$, $R \le y_0 < 0$, can not leave the compact set

$$\left\{(x,y) \in \Omega_{\beta,k} \mid -F(x) + \beta G(x) + k - \frac{1}{\beta} \le y \le 0 \text{ and } a_2 \le x \le 0\right\}$$

through the arc

$$y = -F(x) + \beta G(x) + k - \frac{1}{\beta}, a_2 \leq x \leq 0.$$

Then there is $t_1 > 0$ such that the solution $\gamma(t)$ crosses the $x < 0$ half-axis at a point $\gamma(t_1) = (a_1, 0)$, $a_2 \leq a_1 < 0$, and $y(t) > -F(x(t)) + \beta G(x(t)) + k - \frac{1}{\beta}$, for $0 \leq t \leq t_1$. (See Figure 3.)

In a similar way we prove the following lemmas.

Lemma 2.4 *Suppose there are $a < 0$, $\alpha < 0$ and $k \geq 0$ such that*

$$F(x) \leq \alpha G(x) + k, \quad a \leq x \leq 0.$$

Let $\gamma(t)$ be the soluting of (2) such that $\gamma(0) = (x_0, 0)$, $a \leq x_0 < 0$ and $t_1 > 0$ such that $a \leq x(t) \leq 0$ for $0 \leq t \leq t_1$. Then for $0 \leq t \leq t_1$

$$y(t) < -F(x(t)) + \alpha G(x(t)) + k - \frac{1}{\alpha}.$$

In particular, if $x(t_1) = 0$, then $y(t_1) < k - \frac{1}{\alpha}$.

Lemma 2.5 *Suppose there are $a < 0$, $\alpha \leq 0$, $R < 0$ and $k \geq 0$ such that*

$$F(x) \leq \alpha G(x) + k, \; a \leq x \leq 0 \text{ and } F(a) \leq k - [(R-k)^2 - 2G(a)]^{1/2}.$$

Let $\gamma(t)$ be the solution of (2) with $\gamma(0) = (0, y_0)$, $R \leq y_0 < 0$. Then there is $t_1 > 0$ such that

$$\gamma(t_1) = (a_1, 0), \quad a \leq a_1 < 0.$$

Moreover, if there exists $t_2 > t_1$ such that $a \leq x(t) \leq 0$ for $t_1 \leq t \leq t_2$ and $x(t_2) = 0$ then

$$\gamma(t_2) < k - \frac{1}{\alpha} \text{ if } \alpha > 0 \text{ and } \gamma(t_2) \leq -R + 2k \text{ if } \alpha = 0.$$

Lemma 2.6 *Suppose there are $b > 0$, $\beta < 0$, $R > 0$ and $k \leq 0$ such that*

$$F(x) \geq \beta G(x) + k, \quad 0 \leq x \leq b$$

$$F(b) \geq \beta G(a) + k - \frac{1}{\beta}$$

$$\text{and} \qquad R < k - \frac{1}{\beta}$$

Let $\gamma(t)$ be the solution of (2) with $\gamma(0) = (0, y_0)$, $0 < y_0 \leq R$. Then there is $t_1 > 0$ such that $\gamma(t_1) = (b_1, 0)$, with $0 < b_1 \leq b$.

To close the section we observe that the solutions of (2) do not admit vertical asymptotes (see[1]).

3. Sufficient Condition for Nonexistence of Periodic Solutions

Theorem 3.1 *Suppose there are $a < 0 < b$, $\alpha > 0$ and $k \leq 0$ such that*
(i) $g(x)F(x) > 0$ *for* $a < x < b$ *and* $x \neq 0$;
(ii) $F(a) \leq -[(k - \frac{1}{\alpha})^2 - 2G(a)]^{1/2}$ *and* $F(b) \geq [(k - \frac{1}{\alpha})^2 - 2G(b)]^{1/2}$;
(iii) $F(x) \geq \alpha G(x) + k$ *for* $x > 0$.
Under these conditions the system (2) does not admit non-trivial periodic solution.

Proof. Consider the solution $\gamma(t) = (x(t), y(t))$ starting at $\gamma(0) = (0, y_0)$ with $k - \frac{1}{\alpha} < y_0 < 0$. Suppose there is a smaller $t_2 > 0$ such that $\gamma(t_2) = (0, y_2)$, $y_2 > 0$. From Lemma 2.5 and conditions (i)-(ii) there is $0 < t_1 < t_2$ such that $\gamma(t_1) = (x_1, 0)$, $a \leq x_1 < 0$. It follows that $a \leq x(t) \leq 0$ for $0 \leq t \leq t_2$ and $x(t_2) = 0$. From Lemma 2.5, $y_2 = y(t_2) < \frac{1}{\alpha} - k$. Suppose now there is a smaller $t_4 > t_2$ such that $\gamma(t_4) = (0, y_4)$, $y_4 < 0$. From Lemma 2.2 and conditions (i)-(ii) there is $t_2 < t_3 < t_4$ such that $\gamma(t_3) = (x_3, 0)$, $0 < x_3 \leq b$. It follows that $0 \leq x(t) \leq b$ for $t_2 \leq t \leq t_4$ and $x(t_4) = 0$. From Lemma 2.1 and condition (iii) we have $y_4 = y(t_4) > k - \frac{1}{\alpha}$. From (i) we have

$$\dot{S}_{0,0}(\gamma(t)) < 0 \quad \textit{for } 0 < t < t_4, \quad t \neq t_2.$$

So, $S_{0,0}(\gamma(0)) > S_{0,0}(\gamma(t_4))$ and therefore $\gamma(0) \neq \gamma(t_4)$. It follows that all solution starting at a point $(0, y)$ with $k - \frac{1}{\alpha} < y < 0$ is not periodic.

Consider now the solution $\gamma(t)$ with $\gamma(0) = (x_0, 0)$, $x_0 > 0$ and suppose there is $t_1 > 0$ such that $0 \leq x(t) \leq x_0$ for $0 \leq t \leq t_1$ and $x(t_1) = 0$. From Lemma 2.1 and condition (iii) we have

$$k - \frac{1}{\alpha} < y(t_1) < 0.$$

So, the system (2) does not admit non trivial periodic solution.

We observe that Theorem 1 in [6] is a particular case of our Theorem 3.1.

Remark 3.1 It can easily be verified that the conditions (i), (ii) and (iii) in Theorem 3.1 ensure all solution starting at $(x_0, 0)$, $x_0 > 0$, approaches the origin as $t \to +\infty$.

Remark 3.2 From Lemma 2.4 it follows that the condition (iii) can be replaced by: *there are $\alpha < 0$, and $k \geq 0$ such that*

$$F(x) \leq \alpha G(x) + k \quad \text{for } x < 0.$$

Theorem 3.2 *Suppose there are $\alpha > 0$ and $a < 0$ such that*
(i) $g(x)[F(x) - \alpha G(x)] > 0$ *for* $x \geq a$ and $x \neq 0$;
(ii) $F(a) \leq \alpha G(a) - \frac{1}{\alpha}$.
Under these conditions the system (2) does not admit non-trivial periodic solutions.

Proof. Consider the solution $\gamma(t) = (x(t), y(t))$ with $\gamma(0) = (x_0, 0)$, $x_0 > 0$, and suppose there is $t_1 > 0$ such that $\gamma(t_1) = (x_1, 0)$, $x_1 > 0$. From Lemmas 2.1-2.3 we have

$$y(t) > -F(x(t)) + \alpha G(x(t)) - \frac{1}{\alpha} \qquad \text{and} \qquad x(t) \geq a$$

for $0 \leq t \leq t_1$. Hence and from (i) it follows that for all $t \in [0, t_1]$, with $x(t) \neq 0$, $\dot{S}_{\alpha,0}(\gamma(t)) < 0$. So

$$S_{\alpha,0}(\gamma(0)) > S_{\alpha,0}(\gamma(t_1))$$

and therefore $\gamma(0) \neq \gamma(t_1)$. It follows that the system (2) does not admit non-trivial periodic solution.

Remark 3.3 From Lemmas 2.4-2.6 it follows that the condition (i) and (ii) can be replaced by: *there are $\alpha < 0$ and $b > 0$ such that*

$$g(x)[F(x) - \alpha G(x)] > 0 \quad \text{for } x \leq b \text{ and } x \neq 0 \quad \text{and} \quad F(b) \geq \alpha G(b) - \frac{1}{\alpha}.$$

Remark 3.4 It can be immediately verified that the conditions (i) and (ii) can be replaced by: *there is $\alpha \in \mathbb{R}$ such that*

$$g(x)[F(x) - \alpha G(x)] > 0 \quad \text{for } x \neq 0.$$

Example 3.1 The equation

$$\ddot{x} + (x^5 - x^4 + 3x^2 + 2x)\dot{x} + x = 0$$

does not admit non-trivial periodic solution.

Solution: $F(x) = \frac{x^6}{6} - \frac{x^5}{5} + x^3 + x^2$ and $G(x) = \frac{x^2}{2}$.
For $x \geq -1$ and $x \neq 0$ we have $[F(x) - \alpha G(x)]g(x) > 0$, with $\alpha = 2$. By other hand,

$$F(-1) < \alpha G(-1) - \frac{1}{\alpha}.$$

From Theorem 3.2 the equation does not admit non-trivial periodic solution.

We observe, in the example above,the Theorem 3.1 can not be applied because $F(x) > 0$ for $x < 0$. Also, the theorem 2.1 in [7] and theorem 1 in [8] can not be applied because there are $x_1 > 0$ and $x_2 > 0$ such that $F_e(x_1) > 0$ and $F_e(x_2) < 0$, where $F_e(x) = \int_0^1 f_e(s)ds$ and $f_e(x) = -x^4 + 3x^2$.

Example 3.2 The equation

$$\ddot{x} + (x^3 + 6x^2)\dot{x} + \frac{2x}{(2 + 2x + x^2)^2} = 0$$

does not admit non-trivial periodic solution.

Solution: $F(x) = \frac{x^4}{4} + 2x^3$ and $G(x) = \frac{(x+1)^2}{1+(x+1)^2} - \frac{1}{2} - 2\int_1^{x+1} \frac{1}{(1+u^2)^2} du$
For $\alpha = 1$ and $k = -1$ we have

$$F(x) \geq \alpha G(x) + k \text{ for } x \geq 0 \quad \text{and} \quad F(-1) < -\Big[(k - \frac{1}{\alpha})^2 - 2G(-1)\Big]^{1/2}.$$

We have also

$$g(x)F(x) > 0 \quad \text{for } x \geq -1 \text{ and } x \neq 0.$$

From Theorem 3.1 the equation does not admit non-trivial periodic solution. (Here the condition $F(b) \geq \Big[(k - \frac{1}{\alpha})^2 - 2G(b)\Big]^{1/2}$ is not necessary because $g(x)F(x) > 0$ for all $x \geq -1$ and $x \neq 0$.) The theorems in [7,8] can not be applied because $g(x)$ is not odd.

4. Sufficient Conditions for the Origin to Be Globally Asymptotically Stable

Theorem 4.1 *Suppose the following conditions are verified:*
(i) *There is $\alpha \in \mathbb{R}$ such that*

$$g(x)[F(x) - \alpha G(x)] > 0 \quad \textit{for } x \neq 0;$$

(ii) *There are $k \leq 0$ and $k_1 \geq 0$ such that $F(x) \geq k$ for $x > 0$ and $F(x) \leq k_1$ for $x < 0$.*
(iii) *For all $R > 0$ there are $m < 0 < n$ such that*

$$F(n) \geq k + [(R-k)^2 - 2G(n)]^{1/2} \quad \textit{and} \quad F(m) \leq k_1 - [(R-k_1)^2 - 2G(m)]^{1/2}.$$

Under these conditions the origin is globally asymptotically stable in Liapunov sense.

Proof. Consider the arcs

$$S_{\alpha,0}(x,y) = c \quad \text{with} \quad y \geq -F(x) + \alpha G(x) \tag{14}$$

$$S_{\alpha,0}(x,y) = c \quad \text{with} \quad y \leq -F(x) + \alpha G(x) \tag{15}$$

From hypotheses (i)-(iii) the arc (14) intercepts the $x > 0$ half-axis at $(x_1, 0)$, $x_1 > 0$, and (15) crosses the $x < 0$ half-axis at $(x_2, 0)$, $x_2 < 0$. Let K_c be, $c > 0$, the compact set bounded by the arcs (14), (15) and by the lines $x = x_1$ and $x = x_2$. From (i) we have

$$\dot{S}_{\alpha,0}(x,y) < 0 \quad \text{for } x \neq 0. \tag{16}$$

So, K_c is an invariant set for the system (2). Then the condition (16), by La Salle Theorem, ensures that the origin is asympotically stable and every solution starting at a point in K_c approaches the origin as $t \to +\infty$. It follows that every solution starting at a point in $\Omega_{\alpha,0}$ approaches the origin as $t \to +\infty$. From Lemmas 2.2 and 2.5 and conditions (ii)-(iii) for every solution $\gamma(t)$ of (2) there is t_1 such that $\gamma(t_1) \in \Omega_{\alpha,0}$ So all solution of (2) approaches the origin as $t \to +\infty$. Therefore the origin is globally asymptotically stable.

Remark 4.1 The condition (i) can be replaced by: *There are* $\alpha > 0$ $(\alpha < 0)$, $a < 0$ $(b > 0)$ *such that*

$$g(x)[F(x) - \alpha G(x)] > 0 \quad \text{for } x > a \ (x < b) \text{ and } x \neq 0$$

and

$$F(a) \leq \alpha G(a) - \frac{1}{\alpha}(F(b) \geq \alpha G(b) - \frac{1}{\alpha}.$$

Remark 4.2 The condition (i) can be replaced by: *There are* $a < 0 < b$, $\alpha > 0$ $(\alpha < 0)$, $k \leq 0$ $(k \geq 0)$ *such that*

$$g(x)F(x) > 0 \quad \text{for } a < x < b \text{ and } x \neq 0;$$

$$F(x) \geq \alpha G(x) + k \text{ for } x > 0 \qquad (F(x) \leq \alpha G(x) + k \text{ for } x < 0);$$

$$F(a) \leq -\left[\left(k - \frac{1}{\alpha}\right)^2 - 2G(a)\right]^{1/2}$$

$$\text{and} \qquad F(b) \geq \left[\left(k - \frac{1}{\alpha}\right)^2 - 2G(b)\right]^{1/2}$$

Remark 4.3 Suppose that in Theorem 4.1 the following condition is also verified: (iii) there are $\alpha_1 > 0$, $\alpha_2 < 0$ and $r > 0$ such that

$$F(x) < \alpha_1 G(x) \quad \text{for } 0 < x < r \quad \text{and} \quad F(x) > \alpha_2 G(x) \quad \text{for } -r < x < 0.$$

In this case every non-trivial solution approaches the origin, as $t \to \infty$, in spiral.

We observe that the condition $xF(x) < 0$ for $0 < |x| < \epsilon$ appearing in Theorem 2 in [2] can be replaced by (iii).

Remark 4.4 The condition (iii) is equivalent to the conditions 1.2 and 1.3 appearing in [2].

Example 4.1 For the equation

$$\ddot{x} + (x^4 + 7x^3 + 2x^2 + x)\dot{x} + 5x^3 + x^2 + x = 0$$

the origin is globally asymptotically stable and, for every non-trivial solution $x = x(t)$, $\gamma(t) = (x(t), \dot{x}(t))$ approaches the origin, in spiral, as $t \to +\infty$.

Solution: $F(x) = \frac{x^5}{5} + \frac{7}{4}x^4 + \frac{2}{3}x^3 + \frac{x^2}{2}$ and $G(x) = \frac{5}{4}x^4 + \frac{x^3}{3} + \frac{x^2}{2}$.
We have, for $\alpha = 1$,

$$g(x)[F(x) - \alpha G(x)] > 0 \quad \text{for } x \neq 0$$

and

$$\lim_{x \to +\infty} F(x) = +\infty \quad \text{and} \quad \lim_{x \to -\infty} F(x) = -\infty.$$

We have also there is $r > 0$ such that

$$F(x) < 2G(x) \quad \text{for } 0 < x < r \qquad \text{and} \qquad F(x) > 0 \quad \text{for } -r < x < 0.$$

The conclusion follows from Theorem 4.1 and Remark 4.3.

Example 4.2 Consider again the equation of the Example 3.2:

$$\ddot{x} + (x^3 + 6x^2)\dot{x} + \frac{2x}{(2 + 2x + x^2)^2} = 0.$$

It can be easily verified that the origin is asymptotically stable and, for every non-trivial solution $x = x(t)$ with $\dot{x}(x) = 0$, $\gamma(t) = (x(t, \dot{x}(t))$ approaches the origin as $t \to +\infty$. But the origin is not globally asymptotically stable because there is $a < 0$ such that

$$g(x)[F(x) - G(x)] < 0 \quad \text{for } x \leq a$$

and so the solution $\gamma(t) = (x(t), \dot{x}(t))$, $t \geq 0$, starting at (x_0, y_0) with $x_0 \leq a$ and $y_0 < -F(x_0) + G(x_0) - 1$, does not cross the curve $y = -F(x) + G(x) - 1$.

5. Sufficient Conditions for Existence of Periodic Solutions

Theorem 5.1 *Suppose that*
(i) *the origin is repulsive.*
Suppose also that there are $\alpha > 0$, $k \leq 0$, $k_1 \geq 0$, and $a < 0 < b$ such that:
(ii) $F(x) \leq k_1$ *for* $a \leq x \leq 0$ *and* $F(a) \leq k_1 - \left[\left(k - \frac{1}{\alpha} - k_1\right)^2 - 2G(a)\right]^{1/2}$;
(iii) $F(x) \geq \alpha G(x) + k$ *for* $0 \leq x \leq b$ *and* $F(b) \geq k + \left[\left(\frac{1}{\alpha} + 2k_1 - 2k\right)^2 - 2G(b)\right]^{1/2}$.
Under these conditions the system (2) admits at least one non-trivial periodic solution located between the lines $x = a$ and $x = b$.

Proof. From Lemma 2.5 and hypotheses (i), (ii), the solution staring at the point $(0, k - \frac{1}{\alpha})$ crosses the $y > 0$ half-axis at $(0, y_1)$ with $0 < y_1 \leq -k + \frac{1}{\alpha} + 2k_1$. From

Lemma 2.2 and hypotheses (i), (iii), the solution starting at a point $(0, y_1)$, with $0 < y_1 \leq -k - \frac{1}{\alpha} + 2k_1$ crosses the $y < 0$ half-axis at a point $(0, y_2)$ with $k - \frac{1}{\alpha} < y_2 < 0$. From the Theorem of Poincaré-Bendixon the system (2) admits at least one non-trivial periodic solution. It is clear that this periodic solution is located between the lines $x = a$ and $x = b$.

We observe that the Theorem 3 in [2] is a particular case of the Theorem 5.1.

Remark 5.1 If there are $\alpha \in \mathbb{R}$ and $r > 0$ such that

$$g(x)[F(x) - \alpha G(x)] < 0 \quad \text{for } x < |x| < r$$

then the origin is repulsive. It is enough to observe that the above condition implies

$$\dot{S}_{\alpha,0}(x, y) > 0 \quad \text{for } 0 < |x| < r$$

and for $c > 0$ sufficiently small the level curve $S_{\alpha,0}(x, y) = c$ is closed.

Corollary 5.1. *Suppose that*
(i) *the origin is repulsive.*
Suppose also that there are $\alpha > 0$, $k \leq 0$ and $a < 0 < b$ such that:
(ii) *$F(x) \leq F(a)$ for $a \leq x \leq b$ and $G(a) \geq \frac{1}{2}\left(k - \frac{1}{\alpha} - F(a)\right)^2$;*
(iii) *$F(x) \geq \alpha G(x) + k$ for $0 \leq x \leq b$ and $F(b) \geq k + \left[\left(\frac{1}{\alpha} + 2F(a) - 2k\right)^2 - 2G(b)\right]^{1/2}$.*
Under these conditions the system (2) admits at least one non-trivial periodic solution located between the lines $x = a$ and $x = b$.

Proof. From (ii) there is $a \leq a_1 < 0$ such that $F(x) \leq F(a)$ for $a_1 \leq x \leq 0$ and $2G(a_1) = (k - \frac{1}{\alpha} - F(a))^2$. Now, it is enough to make $k_1 = F(a)$ in Theorem 5.1.

Theorem 5.2 *Suppose that*
(i) *the origin is repulsive.*
Suppose also that these are $\alpha > 0$. $\beta > 0$, $\alpha < 0 < b$, $k \leq 0$ and $k_1 \leq 0$ such that:
(ii) *$F(x) \leq \beta G(x) + k_1$, $\alpha \leq x \leq 0$, $F(a) \leq \beta G(a) + k_1 - \frac{1}{\beta}$ and $k - \frac{1}{\alpha} > k_1 - \frac{1}{\beta}$;*
(iii) *$F(x) \geq \alpha G(x) + k$, $0 \leq x \leq b$ and $F(b) \geq k + [(R - k)^2 - 2G(b)]^{1/2}$ where $R = -k + \frac{1}{\alpha} + 2[\beta G(a) + k_1]$.*
Under these conditions the system (2) admits at least one non-trivial periodic solution located between the line $x = a$ and $x = b$.

Proof. From (ii) we have

$$F(x) \leq \beta G(a) + k - 1, \quad a \leq x \leq 0.$$

From hypotheses and Lemmas 2.2, 2.3 and 2.5 the solution starting at $(0, k - \frac{1}{\alpha})$ crosses again the $y < 0$ half-axis at $(0, y_1)$ with $k - \frac{1}{\alpha} < y_1 < 0$. From theorem of

Poincaré-Bendixon the system (2) admits at least one non-trivial periodic solution. This solution is evidently located between the lines $x = a$ and $x = b$.

Remark 5.2 It can be immediately verified that the condition

$$F(a) \leq \beta G(a) + k_1 - \frac{1}{\beta}$$

can be replaced by

$$S_{\beta,k_1}(0, k - \frac{1}{\alpha}) \leq S_{\beta,k_1}(a, 0).$$

In a similar way it can be proved the following theorems.

Theorem 5.3 *Suppose that*
(i) *the origin repulsive.*
Suppose also that there are $\alpha < 0$ $k \geq 0$, $k_1 \leq 0$ *and* $a < 0 < b$ *such that:*
(ii) $F(x) \geq k_1$ *for* $0 \leq x \leq b$ *and* $F(b) \geq k_1 + \left[(k - \frac{1}{\alpha} - k_1) - 2G(b)\right]^{1/2}$
(iii) $F(x) \leq \alpha G(x) + k$, $a \leq x \leq 0$ *and* $F(a) \leq k - \left[(-\frac{1}{\alpha} - 2k_1 + 2k) - 2G(a)\right]^{1/2}$.
Under these conditions the system (2) *admits at least one non-trivial periodic solution located between the lines* $x = a$ *and* $x = b$.

Corollary 5.2 *Suppose that*
(i) *the origin is repulsive. Suppose also that there are* $\alpha < 0$, $k \geq 0$ *and* $a < 0 < b$ *such that:*
(ii) $F(x) \geq F(b)$ *for* $0 \leq x \leq b$ *and* $G(b) \geq \frac{1}{2}\left(k - \frac{1}{\alpha} - F(b)\right)^2$;
(iii) $F(x) \leq \alpha G(x) + k$ *for* $a \leq x \leq 0$ *and* $F(a) \leq k - \left[\left(-\frac{1}{\alpha} - 2F(b) + 2k\right)^2 - 2G(a)\right]^{1/2}$.
Under these conditions the system (2) *admits at least one non-trivial periodic solution located between the lines* $x = a$ *and* $x = b$.

Theorem 5.4 *Suppose that*
(i) *the origin is repulsive.*
Suppose also that there are $\alpha < 0$, $\beta < 0$, $a < 0 < b$, $k_1 \leq 0$ *such that:*
(ii) $F(x) \geq \beta G(x) + k_1$ *for* $0 \leq x \leq b$ *and* $F(b) \geq \beta G(b) + k_1 - \frac{1}{\beta}$ *and* $k - \frac{1}{\alpha} < k_1 - \frac{1}{\beta}$;
(iii) $F(x) \leq \alpha G(x) + k$ *for* $a \leq x \leq 0$ *and* $F(a) \leq k - \left[(R - k)^2 - 2G(a)\right]^{1/2}$
where $R = -k + \frac{1}{\alpha} + 2[\beta G(b) + k_1]$.
Under these conditions the system (2) *admits at least one non-trivial periodic solution located between the lines* $x = a$ *and* $x = b$.

Remark 5.4 The condition

$$F(b) \geq \beta G(b) + k_1 - \frac{1}{\beta}$$

in Theorem 5.4 can be replaced by

$$S_{\beta,k}\Big(0, k-\frac{1}{\alpha}\Big) \le S_{\beta,k}(b,0).$$

Example 5.1 The equation

$$\ddot{x} + \Big[x^3 + 4x^2 + 3x - \frac{1}{12}\Big]\dot{x} + x = 0$$

admits at least one non-trivial periodic solution.

Solution: $F(x) = \frac{x^4}{4} + \frac{4x^3}{3} + \frac{3x^2}{2} - \frac{x}{12}$ and $G(x) = \frac{x^2}{2}$.
We have

$$\begin{aligned} F(x) \le& F(-1), \quad -1 \le x \le 0, \\ F(x) >& \alpha G(x) + k \quad \text{for all} \quad x > 0 \end{aligned}$$

where $\alpha = 3$ and $k = -\frac{1}{12}$, and

$$G(-1) > \frac{1}{2}\Big[k - \frac{1}{\alpha} - F(-1)\Big]^2.$$

From $\lim_{x\to+\infty} F(x) = +\infty$ it follows that there is $b > 0$ such that

$$F(b) \ge K + \Big[(\frac{1}{\alpha} + 2F(-1) - 2k)^2 - 2G(b)\Big]^{1/2}.$$

The origin is repulsive because there is $r > 0$ such that

$$g(x)[F(x) - G(x)] < 0 \quad \text{for } 0 < |x| < r.$$

From Corollary 5.1 the equation admits at least one non-trivial periodic solution located between the lines $x = -1$ and $x = b$, with $b = \frac{1}{\alpha} + 2F(-1) - 2k$.

Example 5.2 The equation

$$\ddot{x} + (x^3 + 4x^2 - 1)\dot{x} + 4x^3 = 0$$

admits at least one non-trivial periodic solution located between the lines $x = -2$ and $x = 2$.

Solution: $F(x) = \frac{x^4}{4} + \frac{4x^3}{3} - x$ and $G(x) = x^4$.
For $\beta = \frac{1}{4}$ and $k_1 = 1$ we have

$$F(x) \le \beta G(x) + k_1, \quad -2 \le x \le 0 \text{ and } F(-2) \le \beta G(-2) + k_1 - \frac{1}{\beta}.$$

for $\alpha = \frac{1}{2}$ and $k = -\frac{3}{5}$ we have

$$F(x) \geq \alpha G(x) + k, \quad 0 \leq x \leq 2 \text{ and } k - \frac{1}{\alpha} > k_1 - \frac{1}{\beta}$$

. We have also

$$F(2) > k + [(R-k)^2 - 2G(2)]^{1/2}$$

where $R = -k + \frac{1}{\alpha} + 2[\beta G(-2) + k_1]$.

By other hand the origin is evidently repulsive. From Theorem 5.2 the equation admits at least one non-trivial perodic solution located between the lines $x = -2$ and $x = 2$.

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