TAMKANG JOURNAL OF MATHEMATICS
Volume 27, Number 1, Spring 1996

ON A CLASS OF MEROMORPHIC STARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. Let $T_M^*(A, B, z_0)$ denote the class of functions $f(z) = \frac{a}{z} - \sum_{n=1}^{\infty} a_n z^n$ $(a \geq 1, a_n \geq 0)$ regular and univalent in unit disc $U' = \{z : 0 < |z| < 1\}$, satisfying the condition

$$-z\frac{f'(z)}{f(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}, \qquad \text{for } z \in U' \text{ and } w \in E$$

(where E is the class of analytic functions w with $w(0) = 0$ and $|w(z)| \leq 1$), where $-1 \leq A < B \leq 1$, $0 \leq B \leq 1$ and $f(z_0) = \frac{1}{z_0}$ $(0 < z_0 < 1)$. In this paper sharp coefficient estimates, distortion properties and radius of meromorphic convexity for functions in $T_M^*(A, B, z_0)$ have been obtained. We also study integral transforms of functions in $T_M^*(A, B, z_0)$. In the last, it is proved that the class $T_M^*(A, B, z_0)$ is closed under convex linear combinations.

1. Introduction

Let S denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are analytic in $U = \{z : |z| < 1\}$. Denote by $S^*(\rho)$ and $K(\rho)$, $(0 \leq \rho < 1)$ the subclass of functions f in S that satisfy respectively the conditions:

$$\text{Re}[z\frac{f'(z)}{f(z)}] > \rho \text{ and } \text{Re}[1 + \frac{zf''(z)}{f'(z)}] > \rho \quad \text{for } z \in U.$$

Functions in $S^*(\rho)$ and $K(\rho)$ are called starlike functions of order ρ and convex functions of order ρ respectively.

Let T denote the subclass of functions in S of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0. \quad \text{Also set } T^*(\rho) = T \cap S^*(\rho) \text{ and } C(\rho) = T \cap K(\rho).$$

Received April 18, 1993; revised Octember 3, 1994.

The classes $T^*(\rho)$ and $C(\rho)$ possess some very interesting properties and have been studied in detail by Silverman [9,11]. The extreme points for prestar-like functions having negative coefficients have been determined by Silverman and Silvia [12]. In this paper coefficient, distortion and radii of univalence starlikeness and convexity theorem have also been obtained.

Let $T^*(A,B,K)$ be the class of functions $f(z) = a_1 z - \sum_{n=k}^{\infty} |a_n|_z n$ $(a_1 > 0,\ K \geq 2)$ regular and univalent in the unit disc $U = \{z : |z| < 1\}$ and satisfying $|\{(zf'(z)/f(z)) - 1\}/\{A - Bzf'(z)/f(z)\}| < 1$, $z \in U$, Where $-1 \leq B < A \leq 1$ and $-1 \leq B \leq 0$. Let $0 < z_0 < 1$, Kumar [4] denoted by $T_1^*(A,B,K,z_0)$ and $T_2^*(A,B,K,z_0)$, two subclasses of $T^*(A,B,K)$, consisting of functions which satisfy $f(z_0) = z_0$ and $f'(z_0) = 1$ respectively. Kumar [4] has obtained many results including coefficient estimates, distortion and closure theorems and radius of convexity of order $\rho(0 \leq \rho < 1)$ for the classes $T_1^*(A,B,K,z_0)$ and $T_2^*(A,B,K,z_0)$.

Two subclasses obtained by replacing $zf'(z)/f(z)$ by $f'(z)/a_1$ in the definitions of $T_m^*(A,B,K,z_0)$, $m = 1,2$ have been studied by Kumar and Shukla [5].

Let $\sum$ denote the class of functions of the form:

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$$

which are regular in $U' = \{z : 0 < |z| < 1\}$ having a simple pole at the origin. Let $\sum_s$ denote the class of functions in $\sum$ which are univalent in U' and $\sum^*(\rho)$ and $\sum_k(\rho)$ $(0 \leq \rho < 1)$ be the subclasses of functions f in $\sum$ satistying repectively the conditions:

$$R_e\left\{ -z\frac{f'(z)}{f(z)} \right\} > \rho$$

and

$$Re\left[-\left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right] > \rho \qquad \text{for} \quad z \in U'.$$

Functions in $\sum^*(\rho)$ and $\sum_k(\rho)$ are called meromorphically starlike functions of order ρ and meromorphically convex functions of order ρ respectively.

The classes $\sum^*(\rho)$ and $\sum_k(\rho)$ have been extensively studied by Pommerinke [7], Clunie [1], Kazmarski [3], Royster [8] and others.

Since to some extent the work in univalent meromorphic case has paralleled to that of regular univalent case, one is interested to investigate for a class of functions which are regular in U' with simple pole at the origin having properties analogous to those of $T^*(A,B,K)$. To this end we introduce in this section such a class of functions which are regular in U' and which have the properties simillar to those of $T^*(A,B,K)$.

Let T_M^* denote the class of functions $f(z) = \frac{a}{z} - \sum_{n=1}^{\infty} a_n z^n (a \geq 1,\ a_n \geq 0)$, (The condition $a \geq 1$ is necessary, see Nihari [6, ex. 8, p. 238]) regular and univalent in the disc U'. Let $T_M^*(A,B)$ denote the subclass of function in T_M^* satisfying the condition

$$-z\frac{f'(z)}{f(z)} = \frac{1+Aw(z)}{1+Bw(z)}, \text{ for } z \in U', w \in E, \tag{1.1}$$

where $-1 \leq A < B \leq 1, 0 \leq B \leq 1$. Also $T_m^*(A,B,z_0)$ denote the subclass of function in $T_M^*(A,B)$ satisfying $f(z_0) = \frac{1}{z_0}$ (where $0 < z_0 < 1$).

The present chapter is devoted to obtain sharp coefficient estimates, distortion properties and radius of meromorphic convexity for functions in $T_M^*(A,B,z_0)$. We study integral transforms of functions in $T_M^*(A,B,z_0)$. In the last it is shown that the class $T_M^*(A,B,z_0)$ is closed under convex linear combinations.

2. Main Results

First we prove an important theorem which is to be used in next coming theorems.

Theorem 2.1. *Let $f(z) = \frac{1}{z} - \sum_{n=1}^{\infty} |a_n|z^n$ be regular in U' and belongs in $T_M^*(A,B)$ if and only if*

$$\sum_{n=1}^{\infty}\{n(1-B)+1-A\}|a_n| \leq (B-A) \tag{2.1}$$

Proof. Consider the expression

$$H(f,f') = |zf'(z)+f(z)| - |Bzf'(z)+Af(z)|. \tag{2.2}$$

Replacing f and f' by their series expansions we have, for $0 < |z| = r < 1$

$$\begin{aligned} H(f,f') &= \left|\sum_{n=1}^{\infty}(n+1)|a_n|z^n\right| - \left|(A-B)\cdot\frac{1}{z} - \sum_{n=1}^{\infty}(A+Bn)|a_n|z^n\right| \\ &\leq \left|\sum_{n=1}^{\infty}(n+1)|a_n|z^n\right| - \left[\sum_{n=1}^{\infty}(A+Bn)|a_n|z^n - |(A-B)\frac{1}{z}|\right] \\ &= \sum_{n=1}^{\infty}(n+1)|a_n|\,|z|^n - \sum_{n=1}^{\infty}(A+Bn)|a_n|\,|z|^n + (A-B)\frac{1}{|z|} \end{aligned}$$

or

$$rH(f,f') \leq \sum_{n=1}^{\infty}\{n(1-B)+1-A\}|a_n|r^{n+1} + (A-B).$$

Since this holds for all r, $0 < r < 1$, making $r \to 1$, we have

$$H(f,f') \leq \sum_{n=1}^{\infty}\{n(1-B)+1-A\}|a_n| + (A-B) \leq 0,$$

in view of (2.1). From (2.2), we thus have

$$\left|\frac{z\frac{f'(z)}{f(z)}+1}{Bz\frac{f'(z)}{f(z)}+A}\right| \leq 1.$$

Hence $f \in T^*_M(A,B)$.

Conversely, let $f(z) = \frac{1}{z} - \sum\limits_{n=1}^{\infty} |a_n| z^n$, $a_n \geq 0$ is in $T^*_M(A,B)$, i.e.

$$\left|\frac{z\frac{f'(z)}{f(z)}+1}{Bz\frac{f'(z)}{f(z)}+A}\right| \leq 1.$$

or

$$\left|\frac{\sum\limits_{n=1}^{\infty}(n+1)|a_n|z^{n+1}}{(B-A)+\sum\limits_{n=1}^{\infty}(A+Bn)|a_n|z^{n+1}}\right| \leq 1.$$

Since $Re(z) \leq |z|$

$$Re\left\{\frac{\sum\limits_{n=1}^{\infty}(n+1)|a_n|z^{n+1}}{(B-A)+\sum\limits_{n=1}^{\infty}(A+Bn)|a_n|z^{n+1}}\right\} \leq 1.$$

choosing $z = r$ with $0 < r < 1$, we get

$$\frac{\sum\limits_{n=1}^{\infty}(n+1)|a_n|r^{n+1}}{(B-A)+\sum\limits_{n=1}^{\infty}(A+Bn)|a_n|r^{n+1}} \leq 1. \tag{2.3}$$

Let $S(r) = (B-A) + \sum\limits_{n=1}^{\infty}(A+Bn)|a_n|r^{n+1}$, $S(r) \neq 0$ for $0 < r < 1$, $S(r) > 0$ for sufficiently small values of r and $S(r)$ is continuous for $0 < r < 1$. Hence $S(r)$ can not be negative for any value of r such that $0 < r < 1$. Upon clearing the denominator in (2.3) and letting $r \to 1$ we get

$$\sum_{n=1}^{\infty}(n+1)|a_n| \leq (D-A) + \sum_{n=1}^{\infty}(A+Bn)|a_n|,$$

or

$$\sum_{n=1}^{\infty}\{n(1-B)+1-A\}|a_n| \leq B-A.$$

Hence the theorem.

Theorem 2.2. *Let* $f(z) = \frac{a}{z} - \sum_{n=1}^{\infty} |a_n| z^n$ *(where* $a \geq 1$*). If* f *is regular in* U *and satisfies* $f(z_0) = \frac{1}{z_0}$*, then* $f \in T_M^*(A, B, z_0)$ *if and only if*

$$\sum_{n=1}^{\infty} \left[\{n(1-B)+1-A\} - (B-A)z_0^{n+1}\right] |a_n| \leq B - A. \tag{2.4}$$

The result is sharp.

Proof. We know from theorem 2.1 that a function $g(z) = \frac{1}{z} - \sum_{n=1}^{\infty} |b_n| z^n$ regular in U, satisfies

$$\left| \frac{z\frac{g'(z)}{g(z)} + 1}{Bz\frac{g'(z)}{g(z)} + A} \right| < 1, z \in U,$$

if and only if

$$\sum_{n=1}^{\infty} \{n(1-B)+1-A\} |b_n| \leq B - A. \tag{2.5}$$

Applying that result to the function $g(z) = f(z)/a$, we find that f satisfies (1.1) if and only if

$$\sum_{n=1}^{\infty} \{n(1-B)+1-A\} |a_n| \leq (B-A)a.$$

Since $f(z_0) = \frac{1}{z_0}$, we also have from the representation of $f(z)$ that

$$a = 1 + \sum_{n=1}^{\infty} |a_n| z_0^{n+1}. \tag{2.6}$$

Putting this value of a in the above inequality we obtain the required result

$$\sum_{n=1}^{\infty} \left[\{n(1-B)+1-A\} - (B-A)z_0^{n+1}\right] |a_n| \leq B - A.$$

Sharpness follows if we take the extremal function

$$f(z) = \frac{\{n(1-B)+1-A\}\frac{1}{z} - (B-A)z^n}{\{n(1-B)+1-A\} - (B-A)z_0^{n+1}}, n = 1, 2, \ldots. \tag{2.7}$$

Theorem 2.3. $f \in T_M^*$ (A, B, z_0)*, then* f *is meromorphically convex of order* $\delta(0 \leq \delta < 1)$ *in the disc* $|z| < R$*, where*

$$R = \inf_{n>1} \left[\frac{(1-\delta)\{n(1-B)+1-A\}}{n(n+\delta)(B-A)}\right]^{1/(n+1)}.$$

The result is sharp with extremal function (2.7).

Proof. In order to establish the required result, it suffices to show that

$$|2+\frac{zf''(z)}{f'(z)}|\le 1-\delta$$

or

$$|\frac{f'(z)+[zf'(z)]'}{f'(z)}|\le 1-\delta$$

and

$$|\frac{f'(z)+[zf'(z)]'}{f'(z)}|\le \frac{\sum_{n=1}^{\infty}\frac{n(n+1)}{a}|a_n|\ |z|^{n+1}}{1+\sum_{n=1}^{\infty}\frac{n}{a}|a_n|\ |z|^{n+1}}.$$

This will be bounded by $(1-\delta)$ if

$$\sum_{n=1}^{\infty} n(n+\delta)|a_n|\ |z|^{n+1}\le a(1-\delta).$$

Since $a=1+\sum_{n=1}^{\infty}|a_n|z_0^{n+1}$, the above inequality can be written as

$$\sum_{n=1}^{\infty}\frac{[n(n+\delta)|z|^{n+1}-(1-\delta)z_0^{n+1}]}{1-\delta}|a_n|\le 1. \tag{2.8}$$

Also by Theorem 2.2, we have

$$\sum_{n=1}^{\infty}\frac{\{n(1-B)+1-A\}-(B-A)z_0^{n+1}}{(B-A)}|a_n|\le 1.$$

Hence (2.8) will be satisfied if

$$\frac{n(n+\delta)|z|^{n+1}-(1-\delta)z_0^{n+1}}{1-\delta}\le\frac{\{n(1-B)+1-A\}-(B-A)z_0^{n+1}}{(B-A)}$$

or

$$|z|<\left[\frac{(1-\delta)\{n(1-B)+1-A\}}{n(n+\delta)(B-A)}\right]^{1/(n+1)},$$

for each $n=1,2,\ldots$. This completes the proof of theorem.

Theorem 2.4. *If $f\in T_M^*(A,B,z_0)$, then the integral transform*

$$F(z)=c\int_0^1 u^c f(uz)du,\ \textit{for}\ 0<c<\infty$$

is in $T_M(A', B', z_0)$, *where*

$$\frac{1-B'}{B'-A'} \le \frac{(2-A-B)(c+2)-(B-A)c}{2c(B-A)} - \frac{z_0^2}{c}$$

The result is sharp for the extremal function

$$f(z) = \frac{(2-A-B)\frac{1}{z}-(B-A)z}{(2-A-B)-(B-A)z_0^2}.$$

Proof. Suppose $f(z) = \frac{a}{z} - \sum_{n=1}^{\infty} |a_n| z^n \in T_M(A, B, z_0)$, then

$$\begin{aligned} F(z) =& c\int_0^1 u^c\Big[\frac{a}{uz} - \sum_{n=1}^{\infty} |a_n|(u^n z^n)\Big]du \\ =& c\int_0^1 \Big[u^{c-1}\frac{a}{z} - \sum_{n=1}^{\infty} |a_n| z^n u^{n+c}\Big]du \\ =& c\Big[\frac{u^c}{c}\frac{a}{z} - \sum_{n=1}^{\infty} |a_n| z^n \frac{u^{n+c+1}}{(n+c+1)}\Big]_0^1 \\ =& c\Big[\frac{a}{cz} - \sum_{n=1}^{\infty} \frac{|a_n|}{(n+c+1)} z^n\Big] \\ =& \frac{a}{z} - \sum_{n=1}^{\infty} \frac{c}{(n+c+1)} |a_n| z^n. \end{aligned}$$

It is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{[\{n(1-B')+1-A'\}-(B'-A')z_0^{n+1}]}{(B'-A')(n+c+1)} |a_n| \le 1. \tag{2.9}$$

Since $f \in T_M(A, B, z_0)$, it implies that

$$\sum_{n=1}^{\infty} \frac{\{n(1-B)+1-A\}-(B-A)z_0^{n+1}}{(B-A)} |a_n| \le 1.$$

(2.9) will be satisfied if

$$\frac{[\{n(1-B')+1-A'\}-(B'-A')z_0^{n+1}]c}{(B'-A')(n+c+1)} \le \frac{\{n(1-B)+1-A\}-(B-A)z_0^{n+1}}{(B-A)}$$

for each n,

$$\frac{n(1-B')+1-A'}{B'-A'} \le \frac{\{n(1-B)+1-A\}(n+c+1)}{(B-A)c} - \frac{(n+1)}{c} z_0^{n+1},$$

or

$$\frac{1-B'}{B'-A'} \leq \frac{\{n(1-B)+1-A\}(n+c+1)-(B-A)c}{(B-A)(n+1)c} - \frac{1}{c}z_0^{n+1}. \tag{2.10}$$

The right hand side of (2.10) is an increasing function of n, therefore putting $n = 1$ in (2.10) we get

$$\frac{1-B'}{B'-A'} \leq \frac{(2-A-B)(c+2)-(B-A)c}{2c(B-A)} - \frac{z_0^2}{c}.$$

Hence the theorem.

Theorem 2.5. *Let γ be a real number such that $\gamma > 1$. If $f \in T_M^*(A, B, z_0)$, then the function F defined by*

$$F(z) = \frac{(\gamma-1)}{z^\gamma}\int_0^z t^{\gamma-1}f(t)dt$$

also belongs to $T_M^*(A, B, z_0)$.

Proof. Let $f(z) = \frac{a}{z} - \sum_{n=1}^{\infty} |a_n|z^n$. Then from the representation of $F(z)$, it follows that

$$\begin{aligned}
F(z) =& \frac{(\gamma-1)}{z^\gamma}\int_0^z t^{\gamma-1}\Big[\frac{a}{t} - \sum_{n=1}^{\infty}|a_n|t^n\Big]dt \\
=& \frac{\gamma-1}{z^\gamma}\int_0^z \Big[at^{\gamma-2} - \sum_{n=1}^{\infty}|a_n|t^{n+\gamma-1}\Big]dt \\
=& \frac{\gamma-1}{z^\gamma}\Big[a\frac{t^{\gamma-1}}{\gamma-1} - \sum_{n=1}^{\infty}|a_n|\frac{t^{n+\gamma}}{n+\gamma}\Big]_0^z \\
=& \frac{\gamma-1}{z^\gamma}\Big[a\frac{z^{\gamma-1}}{\gamma-1} - \sum_{n=1}^{\infty}\frac{|a_n|}{n+\gamma}z^{n+\gamma}\Big] \\
=& \frac{a}{z} - \sum_{n=1}^{\infty}\frac{\gamma-1}{n+\gamma}|a_n|z^n,
\end{aligned}$$

or

$$F(z) = \frac{a}{z} - \sum_{n=1}^{\infty}|b_n|z^n,$$

where $|b_n| = \frac{\gamma-1}{n+\gamma}|a_n|$. Therefore,

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left[\{n(1-B)+1-A\}-(B-A)z_0^{n+1}\right]|b_n| \\
=&\sum_{n=1}^{\infty}\left[\frac{\gamma-1}{n+\gamma}\right]\left[\{n(1-B)+1-A\}-(B-A)z_0^{n+1}\right]|a_n| \\
\le&\sum_{n=1}^{\infty}\left[\{n(1-B)+1-A\}-(B-A)z_0^{n+1}\right]|a_n| \\
\le&(B-A), \text{ by Theorem 2.2.}
\end{aligned}$$

Hence $F \in T_M^*(A,B,z_0)$. this completes the proof of the theorem.

Theorem 2.6. *Let $f_j(z) = \frac{a_j}{z} - \sum_{n=1}^{\infty}|a_{nj}|z^n$, $j = 1,2,\ldots,m$. If $f_j \in T_M^*(A,B,z_0)$ for each $j = 1,2,\ldots,m$, then the function*

$$h(z) = \frac{b}{z} - \sum_{n=1}^{\infty}|b_n|z^n$$

also belongs to $T_M^*(A,B,z_0)$ where

$$b = \sum_{j=1}^{m}\lambda_j a_j, \quad |b_n| = \sum_{j=1}^{m}\lambda_j|a_{nj}| \qquad (n = 1,2,\ldots,m),$$

$$\lambda_j \ge 0 \text{ and } \sum_{j=1}^{m}\lambda_j = 1$$

Proof. Since $f_j \in T_M^*(A,B,z_0)$, then

$$\sum_{n=1}^{\infty}\left[\{n(1-B)+1-A\}-(B-A)z_0^{n+1}\right]|a_{nj}| \le B-A, \quad j=1,2,\ldots,m.$$

Therefore,

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left[\{n(1-B)+1-A\}-(B-A)z_0^{n+1}\right]|b_n| \\
=&\sum_{n=1}^{\infty}\left[\{n(1-B)+1-A\}-(B-A)z_0^{n+1}\right]\sum_{j=1}^{m}\lambda_j|a_{nj}| \\
=&\sum_{j=1}^{m}\lambda_j\sum_{n=1}^{\infty}\left[\{n(1-B)+1-A\}-(B-A)z_0^{n+1}\right]|a_{nj}| \\
\le&\sum_{j=1}^{m}\lambda_j(B-A) = (B-A).
\end{aligned}$$

Hence by Theorem 2.2, $h \in T_M^*(A, B, z_0)$.

Theorem 2.7. *Let* $f(z) = \frac{1}{z}$ *and*

$$f_n(z) = \frac{\{n(1-B)+1-A\}\frac{1}{z} - (B-A)z^n}{\{n(1-B)+1-A\} - (B-A)z_0^{n+1}},$$

$n = 1, 2, 3, \ldots$ *Then* $h \in T_M^*(A, B, z_0)$ *if and only if it can be expressed in the form*

$$h(z) = \lambda f(z) + \sum_{n=1}^{\infty} \lambda_n f_n(z),$$

where $\lambda \geq 0$ *and* $\lambda + \sum_{n=1}^{\infty} \lambda_n = 1$.

Proof. Let us suppose that

$$\begin{aligned} h(z) =& \lambda f(z) + \sum_{n=1}^{\infty} \lambda_n f_n(z) \\ =& \frac{a}{z} - \sum_{n=1}^{\infty} |a_n| z^n \end{aligned}$$

where

$$a = \lambda + \sum_{n=1}^{\infty} \frac{\{n(1-B)+1-A\}\lambda_n}{\{n(1-B)+1-A\} - (B-A)z_0^{n+1}}$$

and

$$|a_n| + \frac{(B-A)\lambda_n}{\{n(1-B)+1-A\} - (B-A)z_0^{n+1}}, (n = 1, 2, \ldots.).$$

Then, it is easy to see that $f(z_0) = \frac{1}{z_0}$ and the condition (2.4) is satisfied.

Hence $h \in T_M^*(A, B, z_0)$.

Conversely let $h \in T_M^*(A, B, z_0)$, and

$$h(z) = \frac{a}{z} - \sum_{n=1}^{\infty} |a_n| z^n.$$

Then, from (2.4)

$$|a_n| \leq \frac{B-A}{\{n(1-B)+1-A\} - (B-A)z_0^{n+1}}, \quad (n = 1, 2, 3, \ldots).$$

Setting

$$\lambda_n = \left[\frac{\{n(1-B)+1-A\} - (B-A)z_0^{n+1}}{(B-A)}\right] |a_n|$$

and

$$\lambda = 1 - \sum_{n=1}^{\infty} \lambda_n,$$

we have

$$h(z) = \lambda f(z) + \sum_{n=1}^{\infty} \lambda_n f_n(z).$$

This completes the proof of theorem.

Theorem 2.9 *If* $f(z) = \frac{a}{z} - \sum_{n=1}^{\infty} |a_n| z^n \in T_M^*(A, B, z_0)$ *and* $g(z) = \frac{b}{z} - \sum_{n=1}^{\infty} |b_n| z^n$ *with* $|b_n| \le 1$ *for* $n = 1, 2, \ldots$, *then* $f * g \in T_M^*(A, B, z_0)$.

Proof. Let $f(z) = \frac{a}{z} - \sum_{n=1}^{\infty} |a_n| z^n$ and $g(z) = \frac{b}{z} - \sum_{n=1}^{\infty} |b_n| z^n$, then for convolution of functions f and g we can write

$$\begin{aligned}
&\sum_{n=1}^{\infty} \Big[\{n(1-B) + 1 - A\} - (B-A) z_0^{n+1}\Big] |a_n \, b_n| \\
=&\sum_{n=1}^{\infty} \Big[\{n(1-B) + 1 - A\} - (B-A) z_0^{n+1}\Big] |a_n| \, |b_n| \\
\le&\sum_{n=1}^{\infty} \Big[\{n(1-B) + 1 - A\} - (B-A) z_0^{n+1}\Big] |a_n|, \qquad \text{because } |b_n| \le 1. \\
\le&(B-A), \text{ by (2.4).}
\end{aligned}$$

Hence, by Theorem 2.2, $f * g \in T_M^*(A, B, z_0)$.
Note. It will be of interest to find some other convolution results analogous to those of Juneja and Reddy [2].

Acknowledgement

The authors are thankfull to the referee for the helpfull suggestions.

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