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REGION OF STARLIKENESS OF BOUNDED ANALYTIC FUNCTIONS

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Abstract. In this paper we obtain the radius of starlikeness for functions of the type $f(z) = a_1 z + a_2 z^2 + \cdots$ which are analytic and univalent in the unit disk and satisfy $0 < |f(z)| \leq \alpha$ in $0 < |z| < 1$, where α is real.

1. Introduction

Let φ denote the class of functions which are analytic and univalent in the unit disk $\mathcal{E} = \{z : |z| < 1\}$. A function f in φ is said to be starlike of order β if and only if $\mathrm{Re}\,[zf'(z)/f(z)] > \beta \ (0 \leq \beta < 1)$, for z in $\mathcal{E}$. Radius of starlikeness of a function f is the largest r_0, $0 < r_0 < 1$ for which it is starlike of order β in $|z| < r_0$.

2. Main theorem

Let $f(z) = a_1 z + a_2 z^2 + \cdots$ be an anlytic and univalent function in $\mathcal{E}$ and $0 < |f(z)| \leq \alpha$ in $0 < |z| < 1$, where $0 < a_1 < \alpha$ and all other coefficients $a_2, a_3, \ldots$ are complex. Then $f(z)$ is starlike function of order β, in $|z| < r_0$, where r_0 is the smallest positive root of

$$r^3 a_1 \alpha - r^2(a_1^2 \beta + 2\alpha^2) + r(2\beta + 1)a_1 \alpha + \alpha^2 \beta = 0$$

Proof. For $\alpha = a_1$, $f(z) = a_1 z$, the proof is trivial. For general case let us consider

$$F(z) = \{f(z)/\alpha z - a_1/\alpha\}/[1 - a_1 f(z)/\alpha^2 z] \qquad (1)$$

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Clearly $F(z)$ satisfies Schwarz's condition in $\mathcal{E}$. Hence by Schwarz's lemma

$$\left|\frac{f(z)/\alpha z - a_1/\alpha}{1 - a_1 f(z)/\alpha^2 z}\right| \leq |z| \qquad \text{for all } z \text{ in } \mathcal{E}. \tag{2}$$

This on simplification gives us

$$\begin{aligned}&|f(z)/z|^2 - 2a_1\mathrm{Re}(f(z)/z) + a_1^2\\ \leq&\alpha^2|z|^2(1 - [2a_1/\alpha^2]\mathrm{Re}(f(z)/z) + [a_1^2/\alpha^4]|f(z)/z|^2).\end{aligned}$$

Then we get

$$(1 - a_1^2 r^2/\alpha^2)|f(z)/z|^2 - 2(1 - r^2)a_1\mathrm{Re}(f(z)/z) \leq \alpha^2 r^2 - a_1^2,$$

which is equivalent to

$$\left|\frac{f(z)}{z} - \frac{(1 - r^2)\alpha^2 a_1}{\alpha^2 - a_1^2 r_1^2}\right| \leq \frac{\alpha r(\alpha^2 - a_1^2)}{\alpha^2 - a_1^2 r_1^2}. \tag{3}$$

Now let us write

$$G(z) = f(z)/z. \tag{4}$$

Since $f(z)/\alpha$ satisfies the hypothesis of Schwarz's lemma, we have $|G(z)| \leq \alpha$ and from a well known result

$$|G'(z)| \leq \alpha(1 - |G(z)|^2)/(1 - r^2). \tag{5}$$

Differentiation of (4) gives us in light of (5)

$$|f'(z) - f(z)/z| \leq r(\alpha - [1/\alpha]|f(z)/z|^2)/(1 - r^2). \tag{6}$$

This on simplification gives us

$$\begin{aligned}Re\left\{\frac{zf'(z)}{f(z)}\right\} &\geq 1 - \frac{r^2(\alpha - [1/\alpha]|f(z)/z|^2)}{|f(z)|(1 - r^2)}\\ &= \frac{r}{|f(z)|}\left(|f(z)/z| - \frac{r(\alpha - [1/\alpha]|f(z)/z|^2)}{(1 - r^2)}\right)\\ &\geq \frac{r}{|f(z)|}\left(\mathrm{Re}(f(z)/z) - \frac{r(\alpha - [1/\alpha]|f(z)/z|^2)}{(1 - r^2)}\right)\\ &= \frac{r^2}{\alpha|f(z)|(1 - r^2)}\left\{\left|\frac{f(z)}{z}\right|^2 + \frac{\alpha(1 - r^2)}{r}\mathrm{Re}(\frac{f(z)}{z}) - \alpha^2\right\}\\ &\geq \frac{r^2}{\alpha(\alpha)(1 - r^2)}\left\{\left|\frac{f(z)}{z} + \frac{\alpha(1 - r^2)}{2r}\right|^2 - \frac{\alpha^2(1 + r^2)^2}{4r^2}\right\}.\end{aligned}$$

Now for $Re\{zf'(z)/z\} > \beta$, we must have

$$\left\{\left|\frac{f(z)}{z} + \frac{\alpha(1-r^2)}{2r}\right|^2 - \frac{\alpha^2(1+r^2)^2}{4r^2}\right\} > \frac{\beta\alpha^2(1-r^2)}{r^2}. \tag{7}$$

Next we have from (3)

$$\left|\left\{\frac{f(z)}{z} + \frac{\alpha(1-r^2)}{2r}\right\} - \left\{\frac{\alpha^2(1-r^2)a_1}{\alpha^2 - a_1^2r^2} + \frac{\alpha(1-r^2)}{2r}\right\}\right| \le \frac{\alpha r(\alpha^2 - a_1^2)}{\alpha^2 - a_1^2r^2}.$$

Which is equivalent to

$$\begin{aligned}\left|\frac{f(z)}{z} + \frac{\alpha(1-r^2)}{2r}\right| &\ge \frac{\alpha^2(1-r^2)a_1}{\alpha^2 - a_1^2r^2} + \frac{\alpha(1-r^2)}{2r} - \frac{\alpha r(\alpha^2 - a_1^2)}{\alpha^2 - a_1^2r^2} \\ &= \frac{\alpha}{2r(\alpha - a_1r)}(\alpha + a_1r - 3\alpha r^2 + a_1^2r^3).\end{aligned}$$

Finally for $\mathrm{Re}(zf'(z)/z) > \beta$, we must have

$$\frac{\alpha^2}{4r^2(\alpha - a_1r)^2}(\alpha + a_1r - 3\alpha r^2 + a_1r^3) \ge \frac{\beta\alpha^2(1-r)^2}{r^2} + \frac{\alpha^2(1+r^2)^2}{4r^2},$$

consequently, we get

$$(\alpha + a_1r - 3\alpha r^2 + a_1r^3)^2 \ge \{4\beta + 1 - r^2(4\beta - 2) + r^4\}(\alpha - a_1r)^2.$$

This on simplification gives us

$$r^3a_1\alpha - r^2(a_1^2\beta + 2\alpha^2) + r(2\beta + 1)a_1\alpha + \alpha^2\beta = 0.$$

Hence the result follows.

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