TAMKANG JOURNAL OF MATHEMATICS
Volume 26, Number 1, Spring 1995

RINGS WITH A DERIVATION WHOSE IMAGE IS ZERO ON THE ASSOCIATORS

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Abstract. Let R be a nonassociative ring, N,M, L and G the left nucleus,middle nucleus, right nucleus and nucleus respectively. Assume that R is a ring with a derivation d such that $d((R,R,R)) = 0$. It is shown that if R is a simple ring then either R is associative or $d(N \cap L) = 0$; and if R is a prime ring satisfying $Rd(G) \subseteq N$ and $d(G)R \subseteq L$, or $d(G)R + Rd(G) \subseteq M$ then either R is associative or $d(G) = 0$. These partially extend our previous results.

1. Introduction

Let R be a nonassociative ring. We adopt the usual notations for associators and commutators: $(x,y,z) = (xy)z - x(yz)$ and $(x,y) = xy - yx$. We shall denote the left nucleus, middle nucleus, right nucleus and nucleus by N, M, L and G respectively. Thus N, M, L and G consists of all elements n such that $(n,R,R) = 0$, $(R,n,R) = 0$, $(R,R,n) = 0$ and $(n,R,R) = (R,n,R) = (R,R,n) = 0$ respectively. An additive mapping d on R is called a derivation if $d(xy) = d(x)y + xd(y)$ for all x,y in R. R is called semiprime if the only ideal of R which squares to zero is the zero ideal. R is called prime if the product of any two nonzero ideals of R is nonzero. R is called simple if R is the only nonzero ideal of R. Clearly, a prime ring is a semiprime ring. If R is a simple ring, then $R^2 = 0$ or $R^2 = R$; in the former case R is commutative and associative. So, if R is a simple ring then we assume that $R^2 = R$. Thus a simple ring is a prime ring. Recently, Suh [1] proved that if R is a prime ring with a derivation d such that $d(R) \subseteq G$ then either R is associative or $d^3 = 0$. In the notes [2]-[4], we improved and generalized this result. In this note, we prove that if R is a simple ring with a derivation d such that $d((R,R,R)) = 0$ then either R is associative or $d(N \cap L) = 0$. We also show that if

Received August 2, 1993.
1991 *Mathematics Subject Classification.* Primary 17A36.
Key words and phrases. Nonassociative ring, nucleus, derivation, Lie ideal, d-invariant, semiprime ring, prime ring, simple ring.

R is a prime ring with a derivation d such that $d((R,R,R)) = 0$, and $Rd(G) \subseteq N$ and $d(G)R \subseteq L$, or $d(G)R + Rd(G) \subseteq M$, then either R is associative or $d(G) = 0$.

Let R be a ring with a derivation d. Using the definition of d, and by a direct computation we have

$$d((x,y,z)) = (d(x),y,z) + (x,d(y),z) + (x,y,d(z)) \text{ for all } x,y,z \text{ in R.} \qquad (1)$$

Assume that $d(R) \subseteq A$, where A is a subring of R. For all $a \in A$ and $x \in R$, we obtain that $d(ax) = d(a)x + ad(x) \in A$ and $d(xa) = d(x)a + xd(a) \in A$. Since $ad(x) \in A$ and $d(x)a \in A$, these imply $d(a)x \in A$ and $xd(a) \in A$. Thus, $d(A)R + Rd(A) \subseteq A$. On the other hand, if $d(R) \subseteq G$ then $d(G)R + Rd(G) \subseteq G$ and by (1) we have $d((x,y,z)) = 0$ for all x,y,z in R and so $d((R,R,R)) = 0$. A nonempty subset T of R is called d-invariant if $d(T) \subseteq T$. An additive subgroup S of $(R,+)$ is called a Lie ideal of R if $(S,R) \subseteq S$.

2. Results

Let R be a nonassociative ring. In every ring one may verify the Teichmüller identity

$$(wx,y,z) - (w,xy,z) + (w,x,yz) = w(x,y,z) + (w,x,y)z \text{ for all } w,x,y,z \text{ in } R. \qquad (2)$$

Suppose that $n \in N$. Then with $w = n$ in (2) we obtain

$$(nx,y,z) = n(x,y,z) \text{ for all } n \text{ in } N, \text{ and all } x,y,z \text{ in } R. \qquad (3)$$

Assume that $m \in L$. Then with $z = m$ in (2) we get

$$(w,x,ym) = (w,x,y)m \text{ for all } m \text{ in } L, \text{ and all } w,x,y \text{ in } R. \qquad (4)$$

As consequences of (2), (3) and (4), we have that $N, M, L, N \cap M, M \cap L, N \cap L$ and G are associative subrings of R. By (1), we see that all of these subrings are d-invariant.

Definition. The associator ideal I of R is the smallest ideal which contains all associators.

Note that I may be characterized as all finite sums of associators and right (or left) multiples of associators, as a consequence of (2). Hence we have

$$I = \sum (R,R,R) + (R,R,R)R = \sum (R,R,R) + R(R,R,R). \qquad (5)$$

We assume that R has a derivation d which satisfies

$$(*) \qquad d((R,R,R)) = 0.$$

Using (2) and (*), and by the definition of d, we get

$$d(w)(x,y,z) + (w,x,y)d(z) = 0 \text{ for all } w,x,y,z \text{ in } R. \qquad (6)$$

Suppose that $w \in N$. Then applying $d(w) \in N$, (3) and (6), we have $(d(w)x, y, z) = d(w)(x, y, z) = 0$. Thus , $d(w)x \in N$ and $d(N)(R, R, R) = 0$. Using $d(N) \subseteq N$, $d(N)(R, R, R) = 0$ implies $d(N)((R, R, R)R) = 0$. Combining these with (5) yields

$$d(N)R \subseteq N \text{ and } d(N) \cdot I = 0. \tag{7}$$

By symmetry, using $d(L) \subseteq L$, (4), (5) and (6), and as above we obtain

$$Rd(L) \subseteq L \text{ and } I \cdot d(L) = 0. \tag{8}$$

Theorem 1. *If R is a simple ring with a derivation d such that $d((R, R, R)) = 0$, then either R is associative or $d(N \cap L) = 0$.*

Proof. By the simplicity of R, we have that $I = 0$ or $I = R$. If $I = 0$, then R is associative. Assume that $I = R$. Then by (7) and (8) , we obtain $d(N) \cdot R = 0$ and $R \cdot d(L) = 0$. Hence, the ideal B of R generated by $d(N \cap L)$ is $B = \sum d(N \cap L)$. Thus, $B \cdot R = 0$ implies $B = 0$. So, $d(N \cap L) = 0$, as desired.

Lemma 1. *Let R be a ring and C a nonempty subset of G. Then $RC \subseteq N$ if and only if $CR \subseteq M$, and $RC \subseteq M$ if and only if $CR \subseteq L$.*

Proof. Using $C \subseteq G$, and with $x \in C$ in (2) we get

$$(wx, y, z) - (w, xy, z) = 0 \text{ for all } w, y, z \text{ in } R.$$

Thus, this implies $wx \in N$ if and only if $xy \in M$. Hence, $RC \subseteq N$ if and only if $CR \subseteq M$. The proof of the second assertion is similar.

Lemma 2 ([5], Lemma 1). *Let R be a ring and C a nonempty subset of G. If $RC \subseteq N$ and $CR \subseteq L$, or $CR + RC \subseteq M$, then $CR + RC \subseteq M$, and the ideal E of R generated by C is $E = \sum C + CR + RC + R \cdot CR$.*

Proof. Obviously, E is an additive subgroup of $(R, +)$. By the hypotheses and Lemma 1, we have that $C \subseteq G$ and $CR + RC \subseteq M$. Using these we obtain that $RC \cdot R = R \cdot CR$,

$$(R \cdot CR)R = R(CR \cdot R) = R(C \cdot R^2) \subseteq R \cdot CR$$

and

$$R(R \cdot CR) = R(RC \cdot R) = (R \cdot RC)R = (R^2 \cdot C)R \subseteq (R \cdot C)R = R \cdot CR$$

. At this point, we have verified that E is an ideal of R.

Theorem 2. *Let R be a prime ring with a derivation d such that $d((R, R, R)) = 0$. If $Rd(G) \subseteq N$ and $d(G)R \subseteq L$, or $d(G)R + Rd(G) \subseteq M$, then either R is associative or $d(G) = 0$.*

Proof. Applying $d(G) \subseteq G$ and Lemma 2, we see that the ideal F of R generated by $d(G)$ is

$$F = \sum d(G) + d(G)R + Rd(G) + R \cdot d(G)R,$$

and $d(G)R + Rd(G) \subseteq M$.

Since $G \subseteq N$, by (7) we get $d(G) \cdot I = 0$. Applying this, $d(G) \subseteq G$ and $d(G)R + Rd(G) \subseteq M$, we obtain $F \cdot I = 0$. By the primeness of R, this implies either $F = 0$ or $I = 0$. Thus, either $d(G) = 0$ or R is associative, as desired.

Assume that $d(R) \subseteq G$. Then as in the Introduction, we obtain $d((R, R, R)) = 0$ and $d(G)R + Rd(G) \subseteq G$. Hence, our results partially generalize the previous results. Applying the results above and Theorem 2, we have the following generalization of Suh's result.

Corollary 1. *If R is a prime ring with a derivation d such that $d(R) \subseteq G$, then either R is associative or $d^2 = 0$.*

In the course of the proof of Theorem 2, we obtain the

Corollary 2. *Let R be a semiprime ring with a derivation d such that $d((R, R, R)) = 0$ and $d(G) \subseteq I$. If $Rd(G) \subseteq N$ and $d(G)R \subseteq L$, or $d(G)R + Rd(G) \subseteq M$, then $d(G) = 0$.*

Let R be a ring with a derivation d such that $d((R, R, R)) = 0$. If N and L are Lie ideals of R then using these, (7) and (8), we have that $(R, d(N)) \subseteq (R, N) \subseteq N$ and $d(N)R \subseteq N$ imply $Rd(G) \subseteq Rd(N) \subseteq N$, and $(d(L), R) \subseteq (L, R) \subseteq L$ and $Rd(L) \subseteq L$ imply $d(G)R \subseteq d(L)R \subseteq L$ respectively. Thus applying Theorem 2, we obtain

Corollary 3. *Let R be a prime ring with a derivation d such that $d((R, R, R)) = 0$. If N and L are Lie ideals of R, then either R is associative or $d(G) = 0$.*

Corollary 4. *Let R be a semiprime ring with a derivation d such that $d((R, R, R)) = 0$ and $d(G) \subseteq I$. If N and L are Lie ideals of R , then $d(G) = 0$.*

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