TAMKANG JOURNAL OF MATHEMATICS
Volume 26, Number 1, Spring 1995

REMARKS ON SOME RESULTS OF ALZER, YANG AND TENG

JOSIP PEČARIĆ

H.Alzer [1] proved:

Theorem A. *If u and v are non-negative, concave functions defined on* $[0,1]$ *satisfying*

$$\int_0^1 u^{2p}(t)dt = \int_0^1 v^{2q}(t)dt = 1, \quad p > 0, \quad q > 0,$$

then

$$\int_0^1 u^p(t)v^q(t)dt \geq \frac{2\sqrt{(2p+1)(2q+1)}}{(p+1)(q+1)} - 1. \tag{1}$$

G. S. Yang and K. Y. Teng [2] proved:

Theorem B. *Let* $\alpha > -1$*;* $p,q > 0$*. If u and v are nonnegative such that* $u(t^{\frac{1}{1+\alpha}})$ *and* $v(t^{\frac{1}{1+\alpha}})$ *are concave on* $[0,1]$ *then we have*

$$\int_0^1 u^p(t)v^q(t)t^\alpha dt \geq \left[\frac{2}{(p+1)(q+1)} - \frac{1}{\sqrt{(2p+1)(2q+1)}}\right](A+1)^{\frac{p}{A}}(B+1)^{\frac{q}{B}} \cdot (\alpha+1)^{\frac{p}{A}+\frac{q}{B}-1}\left(\int_0^1 u^A(t)t^\alpha dt\right)^{\frac{p}{A}}\left(\int_0^1 v^B(t)t^\alpha dt\right)^{\frac{q}{B}}, \tag{2}$$

for $p \leq A \leq 2p$, $q \leq B \leq 2q$*; and*

$$\int_0^1 u^p(t)v^q(t)t^\alpha dt \geq \left[\frac{2(A+1)^{\frac{p}{A}}(B+1)^{\frac{q}{B}}}{(p+1)(q+1)} - 1\right](\alpha+1)^{\frac{p}{A}+\frac{q}{B}-1} \cdot \left(\int_0^1 u^A(t)t^\alpha dt\right)^{\frac{p}{A}}\left(\int_0^1 v^B(t)t^\alpha dt\right)^{\frac{q}{B}}, \tag{3}$$

for $A \geq 2p$, $B \geq 2q$.

Moreover, a simple consequence of Theorem 5.2 from [3] is:

Received June 30, 1993.

Theorem C. *Let p and q be nonzero real numbers and let u and v be nonnegative concave functions on $[0,1]$. Let $a \geq 1$ and $b \geq 1$. suppose*
(i) $p > 0$, $q > 0$; *or* (ii) $p < 0$, $q < 0$, $p + q > -1$, $ap > -1$, $bq > -1$.
Then

$$\int_0^1 u^p(t)v^q(t)dt \geq D(\int_0^1 u^{pa}(t)dt)^{\frac{1}{a}}(\int_0^1 v^{qb}(t)dt)^{\frac{1}{b}} \tag{4}$$

where

$$D = (1+ap)^{1/a}(1+bq)^{1/b}B(1+p, 1+q). \tag{5}$$

Equality in (4) *occurs if*

$$u(t) = t \quad and \quad v(t) = 1 - t. \tag{6}$$

Remark. For $a = b = 2$, we have that the best constant in Alzer's inequality (i) is

$$\sqrt{(2p+1)(2q+1)}B(1+p, 1+q). \tag{7}$$

Also, the same inequality with the best constant (7) is also valid if we have: (iii) $-1/2 < p < 0$, $-1/2 < q < 0$.

Now, we shall prove the following generalization of Theorem B:

Theorem D. *Let $\alpha > -1$; p and q be nonzero real numbers and let u and v be nonnegative functions such that $u(t^{\frac{1}{(1+\alpha)}})$ and $v(t^{\frac{1}{(1+\alpha)}})$ are concave on $[0,1]$. Suppose:* (iv) $A \geq p > 0$, $B \geq q > 0$; *or* (v) $-1 < A \leq p < 0$, $-1 < B \leq q < 0$, $p + q > -1$. *Then*

$$\int_0^1 u^p(t)v^q(t)t^\alpha dt \geq E(\int_0^1 u^A(t)t^\alpha dt)^{\frac{p}{A}}(\int_0^1 v^B(t)t^\alpha dt)^{\frac{q}{B}} \tag{8}$$

where

$$E = (1+A)^{\frac{p}{A}}(1+B)^{\frac{q}{B}}(\alpha+1)^{\frac{p}{A}+\frac{q}{B}-1}B(1+p, 1+q). \tag{9}$$

Equality in (8) *occurs if*

$$u(t) = t^{1+\alpha} \quad and \quad v(t) = 1 - t^{1+\alpha}. \tag{10}$$

Proof. We should apply Theorem C on concave functions $u(t^{\frac{1}{(1+\alpha)}})$ and $v(t^{\frac{1}{(1+\alpha)}})$, and use substitutions: $ap \to A$, $qb \to B$. Then (4) becomes

$$\int_0^1 u^p(t^{\frac{1}{1+\alpha}})v^q(t^{\frac{1}{1+\alpha}})dt \geq D'(\int_0^1 u^A(t^{\frac{1}{1+\alpha}})dt)^{\frac{p}{A}}(\int_0^1 v(t^{\frac{1}{1+\alpha}})dt)^{\frac{q}{B}} \tag{11}$$

where $D' = (1+A)^{\frac{p}{A}}(1+B)^{\frac{q}{B}}B(1+p,1+q)$.

Now, substitution $t^{\frac{1}{(1+\alpha)}} \to t$, gives (8).

It is easy to check that the conditions $a \geq 1$, $b \geq 1$, (i) and (ii) of Theorem C give our conditions (iv) and (v).

Remark. Note that in Theorem B in the case of positive p and q, the possibility $p \leq A \leq 2p$, $B \geq 2q$ and $q \leq B \leq q$, $A \geq 2q$ were not considered. Of course, our constant is the best possible.

References

[1] H. Alzer, "On an integral inequality of R. Bellman," *Tamkang J. Math.*, 22(1991), 187-191.

[2] G. S. Yang and K. Y. Teng, "A generalization of inverse Schwarz's inequality," *Ibid.*, 23(1992), 117-121.

[3] C. Borell, "Inverse Hölder inequalities in one and several dimensions," *J. Math. Anal. Appl.*, 41(1973), 300-312.

Faculty of Textile-Technology, University of Zagreb, Prilaź Baruna Filipovića 126, 41000 Zagreb, Croatia.