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SOME PROPERTIES OF QUASILINEARITY AND MONOTONICITY FOR HÖLDER'S AND MINKOWSKI'S INEQUALITIES

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Abstract. Some properties of quasilinearity and monotonicity for the well known Hölder's and Minkowski's inequalities for positive real numbers are given.

1. Introduction

Let us begin by displaying the notation and symbols that we shall need

$\mathbb{R}$ = the field of real numbers,
$\mathbb{N}$ = the set of positive integers,
$\mathcal{P}_f(\mathbb{N}) = \{I \subset \mathbb{N} \mid I \text{ is a finite subset of } \mathbb{N}\}$,
$\mathbb{R}_+ = \{r \in \mathbb{R} \mid r > 0\}$,
$E = \{x = (x_i)_{i\in I} \mid x_i \in \mathbb{R}_+, i \in I, I \in \mathcal{P}_f(\mathbb{N})\}$,

$$\|x\|_{m,I,p} := \left(\sum_{i\in I} m_i x_i^p\right)^{1/p}, \quad x, m \in E, \ p \in \mathbb{R}\backslash\{0\},$$

$H(m, I, p, x, y) := \|x\|_{m,I,p}\|y\|_{m,I,q} - \|xy\|_{m,I,1}$,
$M(m, I, p, x, y) := (\|x\|_{m,I,p} + \|y\|_{m,I,p})^p - \|x + y\|^p_{m,I,p}$
where $p \in \mathbb{R}\backslash\{1\}$ and $q = p/(p-1)$ and $m, x, y \in E$, $I \in \mathcal{P}_f(\mathbb{N})$.

Theorem A. *For $p > 1$ with q is as above and $m, x, y \in E$ then Hölder's inequality:*

$$H(m, I, p, x, y) \geq 0 \tag{1.1}$$

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holds. The sign of inequality (1.1) *is reversed for* $p < 1$. *In either case the sign of equality holds if and only if :*

$$x_i^p/x_i^q = x_j^p/x_j^q \text{ for all } i, j \in I.$$

The following result is known in literature as Minkowski's inequality

Theorem B. *For* $p > 1$ *and* m, x, y, I *are as above, the following inequality*

$$M(m, I, p, x, y) \geq 0 \tag{1.2}$$

hlods. The sign of inequality in (1.2) *is reversed for* $p < 1$. *In either case, the sign of equality holds iff:*

$$x_i/y_i = x_j/y_j \text{ for all } i, j \in I.$$

The main aim of this paper is to give some results of quasilinearity and monotonicity for the mappings H and M defined above.

2. The main results

The first result is embodied in the next theorem:

Theorem 2.1. *Let* $x, y \in E$ *and* $p > 1$. *Then:*

(i) *For all* $m, s \in E$ *and* I *a finite part of* $\mathbb{N}$ *one has:*

$$H(m+s, I, p, x, y) \geq H(m, I, p, x, y) + H(s, I, p, x, y) \tag{2.1}$$

i,e., the mapping $H(., I, p, x, y)$ *is superadditive on* E;

(ii) *For all* $m, s \in E$ *with* $m \geq s$, *one has*

$$H(m, I, p, x, y) \geq H(s, I, p, x, y), \tag{2.2}$$

i.e., the mapping $H(\cdot, I, p, x, y)$ *is nondecreasing on* E. *The sign of inequalities* (2.1) *and* (2.2) *are reversed if* $p < 1$.

Proof.

(i) By Hölder's inequality

$$(a^p + b^p)^{1/p}(c^p + d^q)^{1/q} \geq ac + bd \tag{2.3}$$

for all $a, b, c, d \geq 0$, one has

$$\begin{aligned}
&H(m+s, I, p, x, y) \\
=&(\|x\|^p_{m,I,p} + \|x\|^p_{s,I,p})^{1/p}(\|y\|^q_{m,I,q} + \|y\|^q_{s,I,q})^{1/q} - \|xy\|_{m,I,1} - \|xy\|_{s,I,1} \\
\geq&\|x\|_{m,I,p}\|y\|_{m,I,p} + \|x\|_{s,I,p}\|y\|_{s,I,q} - \|xy\|_{m,I,1} - \|xy\|_{s,I,1} \\
=&H(m, I, p, x, y) + H(s, I, p, x, y)
\end{aligned}$$

which proves the inequality (2.1.).

(ii) Suppose that $m \geq s$. Then, by the above property, one has:

$$(a^p + b^p)^{1/p} + (c^p + d^p)^{1/p} \geq ((a+c)^p + (b+d)^p)^{1/p}$$

for all $a, b, c, d \geq 0$, one has :

$$\begin{aligned} &M(m+s, I, p, x, y) \\ =&((\|x\|^p_{m,I,p} + \|x\|^p_{s,I,p})^{1/p} + (\|y\|^p_{m,I,p} + \|y\|^p_{m,I,p})^{1/p})^p - \|x+y\|^p_{m,I,p} - \|x+y\|^p_{s,I,p} \\ \geq&(\|x\|_{m,I,p} + \|y\|_{m,I,p})^p + (\|x\|_{s,I,p} + \|y\|_{s,I,p})^p - \|x+y\|^p_{m,I,p} - \|x+y\|^p_{s,I,p} \\ =&M(m, I, p, x, y) + M(s, I, p, x, y) \end{aligned}$$

which proves the first part of the above theorem.

For the second part, we observe that for $m \geq s$ one has:

$$0 \leq M(m-s, I, p, x, y) \leq M(m, I, p, x, y) - M(s, I, p, x, y)$$

which completes the proof of the theorem.

Finally, we also have:

Theorem 2.4. *Let $m, x, y \in E$ and $p > 1$ $(p < 1)$. Then the mapping $M(m, ., p, x, y)$ is superadditive (subadditive) and nondecreasing (nonincreasing) on $\mathcal{P}_f(\mathbb{N})$.*

The argument is similar to that embodied in the proof of the above theorem and we omit it.

Remark. Similar results can be stated for integrals, but we omit the details.

3. Applications

1. Consider the set $S(1) := \{m \in E | m_i \leq 1, i \in I\}$ where I is a finite part of $\mathbb{N}$. Then one has the bounds:

$$0 \leq sup\{H(m, I, p, x, y) | m \in s(1)\} = (\sum_{i \in I} x_i^p)^{1/p} (\sum_{i \in I} y_i^q)^{1/q} - \sum_{i \in I} x_i y_i$$

and

$$0 \leq \sup\{M(m, I, p, x, y) | m \in S(1)\} = ((\sum_{i \in I} x_i^p)^{1/p} + (\sum_{i \in I} y_i^p)^{1/p})^p - \sum_{i \in I} (x_i + y_i)^p$$

for all $p > 1$ and q is as above.

2. Define the sequences:

$$H_n := (\sum_{i=1}^{n} x_i^p)^{1/p} (\sum_{i=1}^{n} y_i^q)^{1/q} - \sum_{i=1}^{n} x_i y_i, \quad n \in \mathbb{N}$$

and

$$M_n := ((\sum_{i=1}^{n} x_i^p)^{1/p} + (\sum_{i=1}^{n} y_i^p)^{1/p})^p - \sum_{i=1}^{n} (x_i + y_i)^p, \quad n \in \mathbb{N}.$$

Then

$$H_n \geq H_{n-1} \geq \ldots \geq H_2 \geq 0 \text{ and } M_n \geq M_{n-1} \ldots \geq M_2 \geq 0$$

for all $n \in \mathbb{N}$ and $n \geq 2$.

For other results in connection with Hölder's and Minkowski's inequalities, see the papers [1-8] where further references are given.

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