TAMKANG JOURNAL OF MATHEMATICS
Volume 26, Number 1, Sping 1995

COMMUTATIVITY THEOREMS FOR RINGS WITH CONSTRAINTS ON COMMUTATORS

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Abstract. Let R be a left (resp. right) s-unital ring and m be a positive integer. Suppose that for each y in R there exist $f(t), g(t), h(t)$ in $Z[t]$ such that $x^m[x,y] = g(y)[x, y^2f(y)]h(y)$ (resp. $[x,y]x^m = g(y)[x,y^2f(y)]h(y)$) for all x in R. Then R is commutative (and conversely). Finally, the result is extended to the case when the exponent m depends on the choice of x and y.

1. Introduction

Throughout the paper, R will denote an associative ring (may be without unity 1), and for any x, y in R the symbol $[x,y]$ stands for the commutator $xy - yx$. A ring R is called left (resp. right) s-unital if $x \in Rx$ (resp. $x \in xR$) for all x in R, and R is called s-unital if $x \in Rx \cap xR$ for all x in R. As usual $\mathbb{Z}[t]$ is the totality of polynomials in t with coefficients in $\mathbb{Z}$, the ring of integers. We consider the following ring properties:

(P_1) For each y in R there exist $f(t), g(t), h(t)$ in $\mathbb{Z}[t]$ such that $x^m[x,y] = g(y)[x, y^2f(y)]h(y)$ for all x in R, where m is a fixed positive integer.

(P_1^*) For each x, y in R there exist a positive integer m and $f(t), g(t), h(t)$ in $\mathbb{Z}[t]$ such that $x^m[x,y] = g(y)[x,y^2f(y)]h(y)$.

(P_2) For each y in R there exist $f(t), g(t), h(t)$ in $\mathbb{Z}[t]$ such that $[x,y]x^m = g(y)[x, y^2f(y)]h(y)$ for all x in R, where m is a fixed positive integer.

(P_2^*) For each x, y in R there exist a positive integer m and $f(t), g(t), h(t)$ in $\mathbb{Z}[t]$ such that $[x,y]x^m \doteq g(y)[x, y^2f(y)]h(y)$.

(CH) For each x, y in R there exist $p(t), q(t) \in t^2\mathbb{Z}[t]$ such that $[x-p(x), y-q(y)] = 0$.

A well-known theorem of Bell [5] states that if $n > 1$ is a positive integer and R is an n-torsion free ring with unity 1 such that $[x^n, y] = [x, y^n]$ for all x, y in R,

Received September 3, 1993; revised February 22, 1994.
1991 *Mathematics Subject Classification.* 16U80.
Key words and phrases. s-unital rings, factor subrings, polynomial identity, semiprime rings, commutators.

then R is commutative. Recently, Psomopoulos [11] generalized the above result and proved that if R is an n-torsion free ring with unity 1 satisfying the polynomial identity $x^t[x^n, y] = [x, y^m]y^s$ for non-negative integers $m \geq 1, n > 1, t \geq 0, s \geq 0$, then R is commutative. In the present paper our objective is to further extend the later result to rings satisfying condition of the form (P_1) or (P_2). We shall also investigate the commutativity of rings satisfying either of the properties (P_1^*) and (P_2^*) together with the Chacron's condition (CH).

2. Some preliminary results

We being by considering the following types of rings:

(a) $\begin{pmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{pmatrix}$, p a prime.

$(a)_1$ $\begin{pmatrix} GF(p) & GF(p) \\ 0 & 0 \end{pmatrix}$, p a prime.

$(a)_r$ $\begin{pmatrix} 0 & GF(p) \\ 0 & GF(p) \end{pmatrix}$, p a prime.

(b) $M_\sigma(K) = \{(\begin{smallmatrix} \alpha & \beta \\ 0 & \sigma(\alpha) \end{smallmatrix})/\alpha, \beta \in K\}$, where K is a finite field with a non-trivial automorphism σ.

(c) A non-commutative ring with non-zero divisors of zero.

(d) $S = \langle 1 \rangle + T, T$ a non-commutative subring of S such that $T[T,T] = [T,T]T = 0$.

In a paper [12], Streb gave a classification for non-commutative rings, which has been used effectively to obtain a number of commutativity theorems (cf. [8], [9] & [10]). Further, from the proof of [12, Corollary (1)] one can easily observe that if R is a non-commutative left s-unital ring, then there exists a factor subring of R which is of type $(a)_1, (b), (c)$ or (d). This result gives the following lemma which plays the key role in our subsequent study (cf. [9, Meta Theorem]).

Lemma 1. *Let P be a ring property which is inherited by factor subrings. If no rings of type $(a)_1$ (resp. $(a)_r$), $(b), (c)$ or (d) satisfy P, then every left (resp. right) s-unital satisfying P is commutative.*

The proofs of the following lemmas can be found in [7], [8] and [10] respectively.

Lemma 2. *Let f be a polynomial in non-commuting indeterminates $x_1, x_2, \ldots, x_n$ with relatively prime integer coefficients. Then the following are equivalent.*

(i) *Every ring satisfying the polynomial identity $f = 0$ has nil commutator ideal.*
(ii) *Every semi-prime ring satisfying $f = 0$ is commutative.*
(iii) *For every prime $p, (GF(p))_2$ fail to satisfy $f = 0$.*

Lemma 3. *Suppose that a ring R with unity 1 satisfies (CH). If R is* *non-commutative, then there exists a factor subring of R which is of type (a) or (b).*

Lemma 4. *If R is left (resp. right) s-unital and not right (resp. left) s-unital, then R has a factor subring of type $(a)_1$ (resp. $(a)_r$).*

3. Main results

Theorem 1. *Let R be a left s-unital ring satisfying the condition (P_1). Then R is commutative (and conversely).*

In order to establish the above theorem, first we prove the following lemma.

Lemma 5. *Let R be a ring with unity 1 satisfying any one of the conditions (P_1) and (P_2). Then R has nil commutator ideal.*

Proof. Let R satisfy (P_1). Replace x by $1+x$ in (P_1) and subtract (P_1), to get $(1+x)^m[x,y] = x^m[x,y]$. This is a polynomial identity and $x = e_{12} - e_{22}, y = e_{12}$ in $(GF(p))_2$ fail to satisfy this equality. Hence by Lemma 2, R has nil commutator ideal.

Using a similar arguments as above, one can establish the result if R satisfies the condition (P_2).

Proof of Theorem 1. First, consider the ring of type $(a)_1$. If R satisfies (P_1), then in $(GF(p))_2$, p a prime, we see that $e_{11}^m[e_{11}, e_{12}] - g(e_{12})[e_{11}, e_{12}^2 f(e_{12})]h(e_{12}) = e_{12} \neq 0$ for any positive integer m and $f(t), g(t), h(t) \in \mathbb{Z}[t]$. Thus no rings of type $(a)_1$ satisfy (P_1).

Next, consider the ring $M_\sigma(K)$, a ring of type (b). If R satisfies (P_1), then let $x = \begin{pmatrix} a & 0 \\ 0 & \sigma(a) \end{pmatrix} (\sigma(a) \neq a), y = e_{12}$. This yields that

$$x^m[x,y] - g(y)[x, y^2 f(y)]h(y) = a^m(a - \sigma(a))e_{12} \neq 0$$

for all integer m and $f(t), g(t), h(t)$ in $\mathbb{Z}[t]$. Thus R cannot be of type (b).

Further, let R be the ring of type (c). If R satisfies (P_1), then in view of the above, R cannot be of type $(a)_1$. Hence in view of Lemma 4, R is also right s-unital. Thus R is s-unital and by [6, Proposition 1], R has unity 1. Now in view of Lemma 2 and Lemma 5 we see that no rings of type (c) satisfy our assumption.

Finally, suppose that R is a ring of type (d). If R satisfies (P_1), then choose $s, t \in T$ such that $[s,t] \neq 0$. A simple computation shows that

$$[s,t] = (1+s)^m[s+1,t] = g(t)[s+1, t^2 f(t)]h(t) = 0.$$

Hence, we find a contradiction that $[s,t] = 0$. This shows that no rings of type $(a)_1, (b), (c)$ or (d) satisfy (P_1) and in view of Lemma 1, R is commutative.

Corollary 1. *Let R be a left s-unital ring in which for every y in R there exist integers $p = p(y) \geq 0, q = q(y) \geq 0, s = s(y) > 1$ such that $x^m[x,y] = y^P[x,y^s]y^q$ for all x in R, where m is a fixed positive integer. Then R is commutative (and conversely).*

In case, if R satisfies (P_2), then we have the following:

Theorem 2. *Let R be a right s-unital ring satisfying (P_2). Then R is commutative (and conversely).*

Proof. Suppose that R is a ring of type $(a)_r$. If R satisfies (P_2), then in $(GF(p))_2$, p a prime, we find that

$$[e_{22}, e_{12}]e_{22}^m - g(e_{12})[e_{22}, e_{12}^2 f(e_{12})]h(e_{12}) = -e_{12} \neq 0$$

for any positive integer m and $f(t), g(t), h(t) \in \mathbb{Z}[t]$. Accordingly, R cannot be of type $(a)_r$. Further, using similar arguments as used to prove Theorem 1 with necessary variations, it can be shown that no rings of type $(b), (c)$ or (d) satisfy our hypothesis. Hence, in view of Lemma 1, R is commutative.

Corollary 2. *Let R be a right s-unital ring in which for every y in R there exist integers $p = p(y) \geq 0, q = q(y) \geq 0, s = s(y) > 1$ such that $[x,y]x^m = y^p[x,y^s]y^q$ for all x in R, where m is a fixed positive integer. Then R is commutative (and conversely).*

Remark. The following example demonstrates that there are non-commutative left (resp. right) s-unital rings satisfying (P_2) (resp. (P_1)).

Example. Let

$$R_1 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

$$\left(\text{resp. } R_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\} \right)$$

be subring of 2×2 matrices over $GF(2)$. Then for any fixed positive integers $m, p, q, s > 1$, R_1 (resp. R_2) satisfies $[x,y]x^m - y^p[x,y^s]y^q = 0$ (resp. $x^m[x,y] - y^p p[x,y^s]y^q = 0$). However, R_1 (resp. R_2) is a non-commutative left (resp. right) s-unital ring.

A careful scrutiny of the proofs of Theorem 1 and Theorem 2 yields that if R is a ring with unity 1 satisfying either of the property (P_1^*) or (P_2^*), then R has no factor subrings of type (a) or (b). Further, if R satisfies (CH), then in view of Lemma 3, we get the following:

Theorem 3. *Let R be a ring with unity 1 satisfying any one of the conditions (P_1^*) and (P_2^*). Further, if R satisfies (CH), then R is commutative (and conversely).*

Corollary 3. *Suppose that R is a ring with unity 1 in which for every x, y in R there exist integers $m > 0, p \geq 0, q \geq 0, s > 1$ such that either $x^m[x,y] = y^p[x,y^s]y$ or $[x,y]x^m = y^p[x,y^s]y^q$. Further, if R satisfies (CH), then R is commutative.*

The existence of ring R with $R^3 = 0$ rules out the possible generalization of Theorem 3 for argitary rings. However, if R is a left (resp. right) s-unital ring satisfying (P_1^*) (resp. (P_2^*)), then in view of the arguments given in the proofs of Theorem 1 and 2, we see that no rings of type $(a)_1$ (resp.$(a)_r$) satisfy (P_1^*) (resp. (P_2^*)). Hence, Lemma 4 yields that R is also right (resp. left) s-unital. Accordingly, R is s unitial and by [6, Proposition 1], we may assume that R has unity 1, Hence, using Theorem 3, we get the following

Theorem 4. *Let R be a left (resp. right) s-unital ring satisfying (P_1^*) (resp. (P_2^*)). Further, if R satisfies (CH), then R is commutative.*

Acknowledgement

The authors are greatly indebted to the refree for his valuable suggestions.

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